Relativistic scattering theory

Karol Kołodziej

Institute of Physics
University of Silesia, Katowice
http://kk.us.edu.pl

Relativistic scattering

Relativistic scattering on fixed target

Relativistic cross section and decay width

Interaction picture

The interaction picture of quantum mechanics (QM) is used, if the Hamiltonian of the physical system can be decomposed into two parts

$$H = H_0 + V,$$

where H_0 does not explicitly depend on time and has a simple form. Let us define

$$\begin{aligned} |\alpha_I(t)\rangle &\equiv e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\alpha_S(t)\rangle\,, \\ \Omega_I(t) &\equiv e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}. \end{aligned}$$

The interaction picture of quantum mechanics (QM) is used, if the Hamiltonian of the physical system can be decomposed into two parts

$$H = H_0 + V,$$

where H_0 does not explicitly depend on time and has a simple form. Let us define

$$\begin{aligned} |\alpha_I(t)\rangle &\equiv e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\alpha_S(t)\rangle\,, \\ \Omega_I(t) &\equiv e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}. \end{aligned}$$

Note that

$$\begin{aligned} & \langle\alpha_S(t)|\Omega_S|\beta_S(t)\rangle \\ = & \langle\alpha_S(t)|\underbrace{e^{-\frac{i}{\hbar}H_0(t-t_0)}e^{\frac{i}{\hbar}H_0(t-t_0)}}_{I}\Omega_S\underbrace{e^{-\frac{i}{\hbar}H_0(t-t_0)}e^{\frac{i}{\hbar}H_0(t-t_0)}}_{I}|\beta_S(t)\rangle\,, \end{aligned}$$

Interaction picture

The interaction picture of quantum mechanics (QM) is used, if the Hamiltonian of the physical system can be decomposed into two parts

$$H = H_0 + V,$$

where H_0 does not explicitly depend on time and has a simple form. Let us define

$$\begin{aligned} |\alpha_I(t)\rangle &\equiv e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\alpha_S(t)\rangle\,, \\ \Omega_I(t) &\equiv e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}. \end{aligned}$$

Note that

$$\begin{aligned} & \langle\alpha_S(t)|\Omega_S|\beta_S(t)\rangle \\ = \; & \langle\alpha_S(t)|\,\underbrace{e^{-\frac{i}{\hbar}H_0(t-t_0)}e^{\frac{i}{\hbar}H_0(t-t_0)}}_{I}\,\Omega_S\,\underbrace{e^{-\frac{i}{\hbar}H_0(t-t_0)}e^{\frac{i}{\hbar}H_0(t-t_0)}}_{I}\,|\beta_S(t)\rangle\,, \end{aligned}$$

Interaction picture

and taking into account that

$$|\beta_I(t)\rangle = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\beta_S(t)\rangle\,, \qquad \langle\alpha_I(t)| = \langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_{0S}(t-t_0)},$$
$$\Omega_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

we get

$$\begin{aligned} & \langle\alpha_S(t)|\Omega_S|\beta_S(t)\rangle \\ = \ & \underbrace{\langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\langle\alpha_I(t)|}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\Omega_I(t)}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\,|\beta_S(t)\rangle}_{|\beta_I(t)\rangle} \end{aligned}$$

Interaction picture

and taking into account that

$$|\beta_I(t)\rangle = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\beta_S(t)\rangle\,, \qquad \langle\alpha_I(t)| = \langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_{0S}(t-t_0)},$$

$$\Omega_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

we get

$$\langle\alpha_S(t)|\Omega_S|\beta_S(t)\rangle$$

$$= \underbrace{\langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\langle\alpha_I(t)|}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\Omega_I(t)}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\,|\beta_S(t)\rangle}_{|\beta_I(t)\rangle}$$

$$=$$

Interaction picture

and taking into account that

$$|\beta_I(t)\rangle = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\beta_S(t)\rangle\,, \qquad \langle\alpha_I(t)|=\langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_{0S}(t-t_0)},$$
$$\Omega_I(t)=e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

we get

$$\begin{aligned}
& \langle\alpha_S(t)|\Omega_S|\beta_S(t)\rangle \\
= \; & \underbrace{\langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\langle\alpha_I(t)|}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\Omega_I(t)}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\,|\beta_S(t)\rangle}_{|\beta_I(t)\rangle} \\
= \; & \langle\alpha_I(t)|\Omega_I(t)|\beta_I\rangle\,,
\end{aligned}$$

Interaction picture

and taking into account that

$$|\beta_I(t)\rangle = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\beta_S(t)\rangle\,, \qquad \langle\alpha_I(t)| = \langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_{0S}(t-t_0)},$$
$$\Omega_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

we get

$$\begin{aligned} & \langle\alpha_S(t)|\Omega_S|\beta_S(t)\rangle \\ = \; & \underbrace{\langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\langle\alpha_I(t)|}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\Omega_I(t)}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\,|\beta_S(t)\rangle}_{|\beta_I(t)\rangle} \\ = \; & \langle\alpha_I(t)|\Omega_I(t)|\beta_I\rangle\,, \end{aligned}$$

thus the matrix element of an operator remains unchanged.

and taking into account that

$$|\beta_I(t)\rangle = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\beta_S(t)\rangle\,, \qquad \langle\alpha_I(t)| = \langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_{0S}(t-t_0)},$$
$$\Omega_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

we get

$$\begin{aligned} & \langle\alpha_S(t)|\Omega_S|\beta_S(t)\rangle \\ = \; & \underbrace{\langle\alpha_S(t)|\,e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\langle\alpha_I(t)|}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_0(t-t_0)}}_{\Omega_I(t)}\underbrace{e^{\frac{i}{\hbar}H_0(t-t_0)}\,|\beta_S(t)\rangle}_{|\beta_I(t)\rangle} \\ = \; & \langle\alpha_I(t)|\Omega_I(t)|\beta_I\rangle\,, \end{aligned}$$

thus the matrix element of an operator remains unchanged.

Note that as, due to the fact that H_0 need not commute with $H = H_S$, we have

$$H_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \neq H_S,$$

but

$$H_{0I}(t) \quad = \quad e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_{0S}e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} = H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

Interaction picture

Note that as, due to the fact that H_0 need not commute with $H = H_S$, we have

$$H_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)} H_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \neq H_S,$$

but

$$H_{0I}(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} = H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

$$=$$

Note that as, due to the fact that H_0 need not commute with $H = H_S$, we have

$$H_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)} H_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \neq H_S,$$

but

$$\begin{aligned} H_{0I}(t) &= e^{\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} = H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &= H_{0S}. \end{aligned}$$

Note that as, due to the fact that H_0 need not commute with $H = H_S$, we have

$$H_I(t) = e^{\frac{i}{\hbar}H_{0S}(t-t_0)} H_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \neq H_S,$$

but

$$\begin{aligned} H_{0I}(t) &= e^{\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} = H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &= H_{0S}. \end{aligned}$$

Let us calculate

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\alpha_I(t)\rangle \;=\; i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\alpha_S(t)\rangle\right)$$

Let us calculate

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right)$$

$$=$$

Let us calculate

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right)$$

$$= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle$$

Interaction picture

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle \\
&=
\end{aligned}
$$

Interaction picture

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S\left|\alpha_S(t)\right\rangle
\end{aligned}
$$

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S\left|\alpha_S(t)\right\rangle \\
&=
\end{aligned}
$$

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\alpha_I(t)\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}|\alpha_S(t)\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}|\alpha_S(t)\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\alpha_S(t)\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}|\alpha_S(t)\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S|\alpha_S(t)\rangle \\
&= -H_{0S}|\alpha_I(t)\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}(H_{0S}+V_S)|\alpha_S(t)\rangle
\end{aligned}
$$

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\alpha_I(t)\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\alpha_S(t)\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\alpha_S(t)\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\alpha_S(t)\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,|\alpha_S(t)\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S\,|\alpha_S(t)\rangle \\
&= -H_{0S}\,|\alpha_I(t)\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left(H_{0S}+V_S\right)|\alpha_S(t)\rangle \\
&=
\end{aligned}
$$

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left(H_{0S}+V_S\right)\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + H_{0S}\left|\alpha_I(t)\right\rangle
\end{aligned}
$$

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left(H_{0S}+V_S\right)\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + H_{0S}\left|\alpha_I(t)\right\rangle \\
&+
\end{aligned}
$$

Interaction picture

Let us calculate

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left(H_{0S}+V_S\right)\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + H_{0S}\left|\alpha_I(t)\right\rangle \\
&+ \underbrace{e^{\frac{i}{\hbar}H_{0S}(t-t_0)}V_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}}_{V_I(t)}\underbrace{e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle}_{\left|\alpha_I(t)\right\rangle}.
\end{aligned}
$$

Interaction picture

Let us calculate

$$\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle\right) \\
&= i\hbar\frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}H_S\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left(H_{0S}+V_S\right)\left|\alpha_S(t)\right\rangle \\
&= -H_{0S}\left|\alpha_I(t)\right\rangle + H_{0S}\left|\alpha_I(t)\right\rangle \\
&+ \underbrace{e^{\frac{i}{\hbar}H_{0S}(t-t_0)}V_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}}_{V_I(t)}\underbrace{e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\left|\alpha_S(t)\right\rangle}_{\left|\alpha_I(t)\right\rangle}.
\end{aligned}$$

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \left| \alpha_I(t) \right\rangle = V_I(t) \left| \alpha_I(t) \right\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.

Interaction picture

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = V_I(t) |\alpha_I(t)\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.

Let us calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) \quad =$$

Interaction picture

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = V_I(t) |\alpha_I(t)\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.
Let us calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \frac{i}{\hbar} H_{0S}\, e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S\, e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = V_I(t) |\alpha_I(t)\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.

Let us calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \frac{i}{\hbar} H_{0S}\, e^{\frac{i}{\hbar} H_{0S}(t-t_0)} \Omega_S\, e^{-\frac{i}{\hbar} H_{0S}(t-t_0)}$$

$$+$$

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.
Let us calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar}H_{0S}\, e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S\, e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &+ \; e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,\frac{\partial\Omega_S}{\partial t}\, e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}\end{aligned}$$

Interaction picture

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.
Let us calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar}H_{0S}\; e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S\; e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\;\frac{\partial\Omega_S}{\partial t}\; e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &-\end{aligned}$$

Interaction picture

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.
Let us calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar}H_{0S}\, e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S\, e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\,\frac{\partial\Omega_S}{\partial t}\, e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &- \frac{i}{\hbar}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}H_{0S}.\end{aligned}$$

In this way we got the evolution equation of the QM state in an interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = V_I(t) |\alpha_I(t)\rangle .$$

Representation of that kind is useful in particular if $V_I(t)$ contains some small parameter, as e.g. electric charge.
Let us calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar} H_{0S}\, e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S\, e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\, \frac{\partial \Omega_S}{\partial t}\, e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &- \frac{i}{\hbar} e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S}. \end{aligned}$$

Interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$

Interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar} H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar} H_{0S}(t-t_0)}$$

$$+$$

Interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar} H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar} H_{0S}(t-t_0)}$$

$$+ \; e^{\frac{i}{\hbar} H_{0S}(t-t_0)} \frac{\partial \Omega_S}{\partial t} e^{-\frac{i}{\hbar} H_{0S}(t-t_0)}$$

Interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$
$$+ \quad e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\frac{\partial\Omega_S}{\partial t}e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}$$
$$-$$

Interaction picture

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \frac{\partial \Omega_S}{\partial t} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&- \frac{i}{\hbar} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S}
\end{aligned}
$$

Interaction picture

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar}H_{0S}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\frac{\partial\Omega_S}{\partial t}e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\ &- \frac{i}{\hbar}e^{\frac{i}{\hbar}H_{0S}(t-t_0)}\Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)}H_{0S} \\ &=\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar} H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar} H_{0S}(t-t_0)} \\
&+ e^{\frac{i}{\hbar} H_{0S}(t-t_0)} \frac{\partial \Omega_S}{\partial t} e^{-\frac{i}{\hbar} H_{0S}(t-t_0)} \\
&- \frac{i}{\hbar} e^{\frac{i}{\hbar} H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar} H_{0S}(t-t_0)} H_{0S} \\
&= -\frac{1}{i\hbar} H_{0S} \Omega_I + \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar} \Omega_I H_{0S}
\end{aligned}$$

Interaction picture

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \frac{\partial \Omega_S}{\partial t} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&- \frac{i}{\hbar} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S} \\
&= -\frac{1}{i\hbar} H_{0S}\Omega_I + \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\Omega_I H_{0S} \\
&=
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \frac{\partial \Omega_S}{\partial t} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&- \frac{i}{\hbar} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S} \\
&= -\frac{1}{i\hbar} H_{0S}\Omega_I + \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\Omega_I H_{0S} \\
&= \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],
\end{aligned}$$

Interaction picture

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \frac{\partial \Omega_S}{\partial t} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&- \frac{i}{\hbar} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S} \\
&= -\frac{1}{i\hbar} H_{0S}\Omega_I + \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\Omega_I H_{0S} \\
&= \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],
\end{aligned}$$

where we have used the equality $H_{0I} = H_{0S}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) &= \frac{i}{\hbar} H_{0S} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&+ e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \frac{\partial \Omega_S}{\partial t} e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} \\
&- \frac{i}{\hbar} e^{\frac{i}{\hbar}H_{0S}(t-t_0)} \Omega_S e^{-\frac{i}{\hbar}H_{0S}(t-t_0)} H_{0S} \\
&= -\frac{1}{i\hbar} H_{0S}\Omega_I + \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\Omega_I H_{0S} \\
&= \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],
\end{aligned}
$$

where we have used the equality $H_{0I} = H_{0S}$.

Interaction picture

Thus, the evolution equation of the QM state in an interaction picture has the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],$$

where $H_{0I} = H_{0S} = H_0$.

We see that

Thus, the evolution equation of the QM state in an interaction picture has the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \left(\frac{\partial\Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],$$

where $H_{0I} = H_{0S} = H_0$.
We see that

in the Schrödinger picture of QM what evolves are the QM states,

Thus, the evolution equation of the QM state in an interaction picture has the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],$$

where $H_{0I} = H_{0S} = H_0$.
We see that
in the Schrödinger picture of QM what evolves are the QM states,
in the Heisenberg picture evolve dynamical variables,

Thus, the evolution equation of the QM state in an interaction picture has the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \left(\frac{\partial\Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],$$

where $H_{0I} = H_{0S} = H_0$.
We see that
in the Schrödinger picture of QM what evolves are the QM states,
in the Heisenberg picture evolve dynamical variables,
and in the interaction picture evolve both the QM states and dynamical variables.

Thus, the evolution equation of the QM state in an interaction picture has the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_I(t) = \left(\frac{\partial \Omega}{\partial t}\right)_I + \frac{1}{i\hbar}\left[\Omega_I, H_{0I}\right],$$

where $H_{0I} = H_{0S} = H_0$.
We see that
in the Schrödinger picture of QM what evolves are the QM states,
in the Heisenberg picture evolve dynamical variables,
and in the interaction picture evolve both the QM states and dynamical variables.

Evolution operator

If $V_I(t)$ contains some small parameter, as e.g. electric charge, then the evolution equation of the QM state in the interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = V_I(t) |\alpha_I(t)\rangle$$

can be considered as the starting point of the perturbative expansion.

Evolution operator

If $V_I(t)$ contains some small parameter, as e.g. electric charge, then the evolution equation of the QM state in the interaction picture

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle$$

can be considered as the starting point of the perturbative expansion.

Define the time evolution operator in the interaction picture $U_I(t', t)$.

$$\left|\alpha_I(t')\right\rangle = U_I(t', t)\left|\alpha_I(t)\right\rangle,$$

Evolution operator

If $V_I(t)$ contains some small parameter, as e.g. electric charge, then the evolution equation of the QM state in the interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = V_I(t) |\alpha_I(t)\rangle$$

can be considered as the starting point of the perturbative expansion.

Define the time evolution operator in the interaction picture $U_I(t', t)$.

$$|\alpha_I(t')\rangle = U_I(t', t) |\alpha_I(t)\rangle , \qquad U_I(t, t) = 1.$$

Evolution operator

If $V_I(t)$ contains some small parameter, as e.g. electric charge, then the evolution equation of the QM state in the interaction picture

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle$$

can be considered as the starting point of the perturbative expansion.

Define the time evolution operator in the interaction picture $U_I(t',t)$.

$$\left|\alpha_I(t')\right\rangle = U_I(t',t)\left|\alpha_I(t)\right\rangle, \qquad U_I(t,t) = 1.$$

The evolution operator $U_I(t',t)$ satisfies the same equation as the state vector $\left|\alpha_I(t)\right\rangle$.

Evolution operator

If $V_I(t)$ contains some small parameter, as e.g. electric charge, then the evolution equation of the QM state in the interaction picture

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = V_I(t) |\alpha_I(t)\rangle$$

can be considered as the starting point of the perturbative expansion.

Define the time evolution operator in the interaction picture $U_I(t', t)$.

$$|\alpha_I(t')\rangle = U_I(t', t) |\alpha_I(t)\rangle , \qquad U_I(t, t) = 1.$$

The evolution operator $U_I(t', t)$ satisfies the same equation as the state vector $|\alpha_I(t)\rangle$.

Perturbation expansion

Indeed, let us calculate

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = i\hbar \frac{\mathrm{d}}{\mathrm{d}t} U_I(t, t_0) |\alpha_I(t_0)\rangle =$$

Perturbation expansion

Indeed, let us calculate

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = i\hbar \frac{\mathrm{d}}{\mathrm{d}t} U_I(t, t_0) |\alpha_I(t_0)\rangle = V_I(t) |\alpha_I(t)\rangle$$

Perturbation expansion

Indeed, let us calculate

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = i\hbar \frac{\mathrm{d}}{\mathrm{d}t} U_I(t, t_0) |\alpha_I(t_0)\rangle = V_I(t) |\alpha_I(t)\rangle$$

$$=$$

Indeed, let us calculate

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_I(t,t_0)\left|\alpha_I(t_0)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle$$
$$= V_I(t)U_I(t,t_0)\left|\alpha_I(t_0)\right\rangle .$$

Perturbation expansion

Indeed, let us calculate

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\alpha_I(t)\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_I(t,t_0)|\alpha_I(t_0)\rangle = V_I(t)|\alpha_I(t)\rangle$$
$$= V_I(t)U_I(t,t_0)|\alpha_I(t_0)\rangle .$$

If we compare coefficients of the arbitrary chosen initial state $|\alpha_I(t_0)\rangle$ we will get

Perturbation expansion

Indeed, let us calculate

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_I(t,t_0)\left|\alpha_I(t_0)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle \\ &= V_I(t)U_I(t,t_0)\left|\alpha_I(t_0)\right\rangle . \end{aligned}$$

If we compare coefficients of the arbitrary chosen initial state $\left|\alpha_I(t_0)\right\rangle$ we will get

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_I(t,t_0) = V_I(t)U_I(t,t_0).$$

Perturbation expansion

Indeed, let us calculate

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\alpha_I(t)\rangle = i\hbar \frac{\mathrm{d}}{\mathrm{d}t} U_I(t, t_0) |\alpha_I(t_0)\rangle = V_I(t) |\alpha_I(t)\rangle$$
$$= V_I(t) U_I(t, t_0) |\alpha_I(t_0)\rangle .$$

If we compare coefficients of the arbitrary chosen initial state $|\alpha_I(t_0)\rangle$ we will get

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} U_I(t, t_0) = V_I(t) U_I(t, t_0).$$

Let's integrate both sides of this equation

$$i\hbar \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t'} U_I(t', t_0) \mathrm{d}t' = \int_{t_0}^{t} V_I(t') U_I(t', t_0) \mathrm{d}t'.$$

Indeed, let us calculate

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_I(t)\right\rangle &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_I(t,t_0)\left|\alpha_I(t_0)\right\rangle = V_I(t)\left|\alpha_I(t)\right\rangle \\ &= V_I(t)U_I(t,t_0)\left|\alpha_I(t_0)\right\rangle . \end{aligned}$$

If we compare coefficients of the arbitrary chosen initial state $\left|\alpha_I(t_0)\right\rangle$ we will get

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_I(t,t_0) = V_I(t)U_I(t,t_0).$$

Let's integrate both sides of this equation

$$i\hbar\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}t'}U_I(t',t_0)\mathrm{d}t' = \int_{t_0}^{t}V_I(t')U_I(t',t_0)\mathrm{d}t'.$$

Perturbation expansion

$$i\hbar\left(U_I(t,t_0)-U_I(t_0,t_0)\right)=\int\limits_{t_0}^{t}V_I(t')U_I(t',t_0)\mathrm{d}t'.$$

After simple modifications and using the initial condition $U_I(t_0,t_0)=1$ we obtain the following integral equation for the time evolution operator in the interaction picture

$$U_I(t,t_0)=1-\frac{i}{\hbar}\int\limits_{t_0}^{t}V_I(t')U_I(t',t_0)\mathrm{d}t'.$$

$$i\hbar\left(U_I(t,t_0)-U_I(t_0,t_0)\right)=\int_{t_0}^{t} V_I(t')U_I(t',t_0)\mathrm{d}t'.$$

After simple modifications and using the initial condition $U_I(t_0,t_0)=1$ we obtain the following integral equation for the time evolution operator in the interaction picture

$$U_I(t,t_0)=1-\frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t',t_0)\mathrm{d}t'.$$

This equation can be iterated.

$$i\hbar\left(U_I(t,t_0) - U_I(t_0,t_0)\right) = \int\limits_{t_0}^{t} V_I(t')U_I(t',t_0)\mathrm{d}t'.$$

After simple modifications and using the initial condition $U_I(t_0,t_0) = 1$ we obtain the following integral equation for the time evolution operator in the interaction picture

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int\limits_{t_0}^{t} V_I(t')U_I(t',t_0)\mathrm{d}t'.$$

This equation can be iterated.

Perturbation expansion

Assume $t \geqslant t_1 \geqslant t_0$ and iterate for the first time

$$U_I(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} V_I(t_1) \left(1 - \frac{i}{\hbar} \int_{t_0}^{t_1} V_I(t') U_I(t', t_0) \mathrm{d}t' \right) \mathrm{d}t_1$$

Perturbation expansion

Assume $t \geqslant t_1 \geqslant t_0$ and iterate for the first time

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\left(1 - \frac{i}{\hbar}\int_{t_0}^{t_1} V_I(t')U_I(t',t_0)\mathrm{d}t'\right)\mathrm{d}t_1$$

$=$

Perturbation expansion

Assume $t \geqslant t_1 \geqslant t_0$ and iterate for the first time

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\left(1 - \frac{i}{\hbar}\int_{t_0}^{t_1} V_I(t')U_I(t',t_0)\mathrm{d}t'\right)\mathrm{d}t_1$$

$$= \quad 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\mathrm{d}t_1 + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t_1} V_I(t_1)V_I(t')U_I(t',t_0)\mathrm{d}t'\mathrm{d}t_1$$

Perturbation expansion

Assume $t \geqslant t_1 \geqslant t_0$ and iterate for the first time

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\left(1 - \frac{i}{\hbar}\int_{t_0}^{t_1} V_I(t')U_I(t',t_0)\mathrm{d}t'\right)\mathrm{d}t_1$$

$$= \quad 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\mathrm{d}t_1 + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t_1} V_I(t_1)V_I(t')U_I(t',t_0)\mathrm{d}t'\mathrm{d}t_1$$

In the second iteration, assuming $t \geqslant t_1 \geqslant t_2 \geqslant t_0$, we will get

$$U_I(t,t_0) \quad = \quad 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\mathrm{d}t_1 + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t_1} V_I(t_1)V_I(t_2)\Big(1$$

$$- \quad \frac{i}{\hbar}\int_{t_0}^{t_2} V_I(t')U_I(t',t_0)\mathrm{d}t'\Big)\mathrm{d}t_2\mathrm{d}t_1.$$

Perturbation expansion

Assume $t \geqslant t_1 \geqslant t_0$ and iterate for the first time

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\left(1 - \frac{i}{\hbar}\int_{t_0}^{t_1} V_I(t')U_I(t',t_0)\mathrm{d}t'\right)\mathrm{d}t_1$$

$$= \quad 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\mathrm{d}t_1 + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t_1} V_I(t_1)V_I(t')U_I(t',t_0)\mathrm{d}t'\mathrm{d}t_1$$

In the second iteration, assuming $t \geqslant t_1 \geqslant t_2 \geqslant t_0$, we will get

$$\begin{aligned} U_I(t,t_0) \quad &= \quad 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t_1)\mathrm{d}t_1 + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t_1} V_I(t_1)V_I(t_2)\Big(1 \\ &- \quad \frac{i}{\hbar}\int_{t_0}^{t_2} V_I(t')U_I(t',t_0)\mathrm{d}t'\Big)\mathrm{d}t_2\mathrm{d}t_1. \end{aligned}$$

By repeating this procedure we will obtain the following formula for the perturbative expansion of the evolution operator

$$U_I(t, t_0) = 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 ... \int_{t_0}^{t_{n-1}} \mathrm{d}t_n V_I(t_1) V_I(t_2) ... V_I(t_n),$$

where in the n-th iteration we have assumed the following time order $t \geqslant t_1 \geqslant t_2 \geqslant ... \geqslant t_n \geqslant t_0$.

Let us introduce the time ordered product of operators

$$T\left[V_I(t_1) V_I(t_2) ... V_I(t_n)\right] \equiv \begin{cases} V_I(t_1) V_I(t_2) ... V_I(t_n), & \text{for } t_1 \geqslant t_2 \geqslant ... \geqslant t_{n-1} \geqslant t_n, \\ 0, & \text{in other cases.} \end{cases}$$

By repeating this procedure we will obtain the following formula for the perturbative expansion of the evolution operator

$$U_I(t,t_0) = 1 + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \dots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n V_I(t_1)V_I(t_2)\dots V_I(t_n),$$

where in the n-th iteration we have assumed the following time order $t \geqslant t_1 \geqslant t_2 \geqslant \dots \geqslant t_n \geqslant t_0$.

Let us introduce the time ordered product of operators

$$T\left[V_I(t_1)V_I(t_2)\dots V_I(t_n)\right] \equiv \begin{cases} V_I(t_1)V_I(t_2)\dots V_I(t_n), & \text{for } t_1 \geqslant t_2 \geqslant \dots \geqslant t_{n-1} \geqslant t_n, \\ 0, & \text{in other cases.} \end{cases}$$

Perturbation expansion

Then the perturbative expansion of the evolution operator takes the form

$$U_I(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \int\limits_{t_0}^{t} \mathrm{d}t_1 \int\limits_{t_0}^{t} \mathrm{d}t_2 ... \int\limits_{t_0}^{t} \mathrm{d}t_n T\left[V_I(t_1)V_I(t_2)...V_I(t_n)\right],$$

where all the upper integration limits are equal.

Exercise. Justify the $\frac{1}{n!}$ factor in the above expression.

Then the perturbative expansion of the evolution operator takes the form

$$U_I(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 ... \int_{t_0}^{t} \mathrm{d}t_n \, T\left[V_I(t_1)V_I(t_2)...V_I(t_n)\right],$$

where all the upper integration limits are equal.

Exercise. Justify the $\frac{1}{n!}$ factor in the above expression.

In the quantum field theory, one usually uses the interaction Hamiltonian density instead of the potential $V_I(t)$ which is defined as

$$V_I(t) = \int \mathrm{d}^3x \, \mathcal{H}_I(x),$$

where one integrates over the full 3-dimensional hyper surface of $t = \mathrm{const}$ in the 4-dimensional Minkowski's space time.

Perturbation expansion

Then the perturbative expansion of the evolution operator takes the form

$$U_I(t, t_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 \ldots \int_{t_0}^{t} \mathrm{d}t_n \, T\left[V_I(t_1) V_I(t_2) \ldots V_I(t_n)\right],$$

where all the upper integration limits are equal.

Exercise. Justify the $\frac{1}{n!}$ factor in the above expression.

In the quantum field theory, one usually uses the interaction Hamiltonian density instead of the potential $V_I(t)$ which is defined as

$$V_I(t) = \int \mathrm{d}^3x \, \mathcal{H}_I(x),$$

where one integrates over the full 3-dimensional hyper surface of $t = \mathrm{const}$ in the 4-dimensional Minkowski's space time.

Thus the perturbative expansion of the evolution operator can be written as

$$U_I(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 ... \int \mathrm{d}^4x_n T\left[\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)...\mathcal{H}_I(x_n)\right].$$

If $\mathcal{H}_I(x)$ contains a small parameter, then it is usually enough to calculate a few lowest order terms of the expansion series of the operator $U_I(t,t_0)$, e.g. in quantum electrodynamics (QED)

$$\mathcal{H}_I(x) = -e\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x).$$

Perturbation expansion

Thus the perturbative expansion of the evolution operator can be written as

$$U_I(t,t_0) = 1$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n\int \mathrm{d}^4x_1\int \mathrm{d}^4x_2 ... \int \mathrm{d}^4x_n T\left[\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)...\mathcal{H}_I(x_n)\right].$$

If $\mathcal{H}_I(x)$ contains a small parameter, then it is usually enough to calculate a few lowest order terms of the expansion series of the operator $U_I(t,t_0)$, e.g. in quantum electrodynamics (QED)

$$\mathcal{H}_I(x) = -e\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x).$$

Perturbative expansion of the evolution operator is extensively used in the scattering theory, where the scattering operator S can be defined as

$$S = U_I(-\infty, +\infty).$$

Scattering operator

Perturbative expansion of the evolution operator is extensively used in the scattering theory, where the scattering operator S can be defined as

$$S = U_I(-\infty, +\infty).$$

Long before the scattering, i.e. for $t \to -\infty$, and long after the scattering, i.e. for $t \to +\infty$, the time evolution of the QM system is described by the Hamiltonian H_0 and we have to do with the asymptotically free states.

Perturbative expansion of the evolution operator is extensively used in the scattering theory, where the scattering operator S can be defined as

$$S = U_I(-\infty, +\infty).$$

Long before the scattering, i.e. for $t \to -\infty$, and long after the scattering, i.e. for $t \to +\infty$, the time evolution of the QM system is described by the Hamiltonian H_0 and we have to do with the asymptotically free states.

In practice it is enough to assume that the time before and after scattering is much longer than the time of interaction of the projectile with the scattering centre.

Perturbative expansion of the evolution operator is extensively used in the scattering theory, where the scattering operator S can be defined as

$$S = U_I(-\infty, +\infty).$$

Long before the scattering, i.e. for $t \to -\infty$, and long after the scattering, i.e. for $t \to +\infty$, the time evolution of the QM system is described by the Hamiltonian H_0 and we have to do with the asymptotically free states.
In practice it is enough to assume that the time before and after scattering is much longer than the time of interaction of the projectile with the scattering centre.

Scattering operator

According to our our definition, state $|\psi(+\infty)\rangle$ of the QM system for $t \to +\infty$ is related to the asymptotically free initial state $|i\rangle \equiv |\psi(-\infty)\rangle$ for $t \to -\infty$ through the equation

$$|\psi(+\infty)\rangle = S\,|i\rangle\,,$$

where

$$H_0\,|i\rangle = E_i\,|i\rangle\,.$$

Scattering operator

According to our our definition, state $|\psi(+\infty)\rangle$ of the QM system for $t \to +\infty$ is related to the asymptotically free initial state $|i\rangle \equiv |\psi(-\infty)\rangle$ for $t \to -\infty$ through the equation

$$|\psi(+\infty)\rangle = S\,|i\rangle\,,$$

where

$$H_0\,|i\rangle = E_i\,|i\rangle\,.$$

In the result of scattering the QM system can go to any QM state of the full spectrum of H_0:

$$H_0\,|f\rangle = E_f\,|f\rangle\,.$$

Scattering operator

According to our our definition, state $|\psi(+\infty)\rangle$ of the QM system for $t \to +\infty$ is related to the asymptotically free initial state $|i\rangle \equiv |\psi(-\infty)\rangle$ for $t \to -\infty$ through the equation

$$|\psi(+\infty)\rangle = S\,|i\rangle\,,$$

where

$$H_0\,|i\rangle = E_i\,|i\rangle\,.$$

In the result of scattering the QM system can go to any QM state of the full spectrum of H_0:

$$H_0\,|f\rangle = E_f\,|f\rangle\,.$$

The probability amplitude of finding the QM system in the final state $|f\rangle$ is given by

$$\langle f|\psi(+\infty)\rangle = \langle f|S|i\rangle = S_{fi},$$

thus it is given by the matrix element of the operator S, which due to this is often referred to as the scattering matrix.

Scattering operator

According to our our definition, state $|\psi(+\infty)\rangle$ of the QM system for $t \to +\infty$ is related to the asymptotically free initial state $|i\rangle \equiv |\psi(-\infty)\rangle$ for $t \to -\infty$ through the equation

$$|\psi(+\infty)\rangle = S\,|i\rangle\,,$$

where

$$H_0\,|i\rangle = E_i\,|i\rangle\,.$$

In the result of scattering the QM system can go to any QM state of the full spectrum of H_0:

$$H_0\,|f\rangle = E_f\,|f\rangle\,.$$

The probability amplitude of finding the QM system in the final state $|f\rangle$ is given by

$$\langle f|\psi(+\infty)\rangle = \langle f|S|i\rangle = S_{fi},$$

thus it is given by the matrix element of the operator S, which due to this is often referred to as the scattering matrix.

The corresponding probability density is given by

$$|\langle f|\psi(+\infty)\rangle|^2 .$$

We assume that the asymptotic states $|f\rangle$ and $|i\rangle$ are normalized to 1.

If we neglect bound states, which have a very small probability to be formed for high energy projectiles, then we can also assume that the *exact* states $|\psi(t)\rangle$ are normalized to 1, i.e.

$$\langle\psi(+\infty)|\psi(+\infty)\rangle = 1.$$

Scattering operator

The corresponding probability density is given by

$$|\langle f|\psi(+\infty)\rangle|^2 .$$

We assume that the asymptotic states $|f\rangle$ and $|i\rangle$ are normalized to 1.

If we neglect bound states, which have a very small probability to be formed for high energy projectiles, then we can also assume that the *exact* states $|\psi(t)\rangle$ are normalized to 1, i.e.

$$\langle\psi(+\infty)|\psi(+\infty)\rangle = 1.$$

Now, we can decompose

$$|\psi(+\infty)\rangle = \sum_f |f\rangle \langle f|\psi(+\infty)\rangle = \sum_f |f\rangle S_{fi},$$

where actually the integral should be used instead of the sum.

The corresponding probability density is given by

$$|\langle f|\psi(+\infty)\rangle|^2 .$$

We assume that the asymptotic states $|f\rangle$ and $|i\rangle$ are normalized to 1.
If we neglect bound states, which have a very small probability to be formed for high energy projectiles, then we can also assume that the *exact* states $|\psi(t)\rangle$ are normalized to 1, i.e.

$$\langle\psi(+\infty)|\psi(+\infty)\rangle = 1.$$

Now, we can decompose

$$|\psi(+\infty)\rangle = \sum_f |f\rangle \langle f|\psi(+\infty)\rangle = \sum_f |f\rangle S_{fi},$$

where actually the integral should be used instead of the sum.
Hence

$$\langle\psi(+\infty)| = \sum_f \langle f| S^*_{fi}.$$

Scattering operator

The corresponding probability density is given by

$$|\langle f|\psi(+\infty)\rangle|^2 .$$

We assume that the asymptotic states $|f\rangle$ and $|i\rangle$ are normalized to 1.
If we neglect bound states, which have a very small probability to be formed for high energy projectiles, then we can also assume that the *exact* states $|\psi(t)\rangle$ are normalized to 1, i.e.

$$\langle\psi(+\infty)|\psi(+\infty)\rangle = 1.$$

Now, we can decompose

$$|\psi(+\infty)\rangle = \sum_f |f\rangle \langle f|\psi(+\infty)\rangle = \sum_f |f\rangle S_{fi},$$

where actually the integral should be used instead of the sum.
Hence

$$\langle\psi(+\infty)| = \sum_f \langle f| \, S^*_{fi}.$$

Scattering operator

Now we can use the normalisation condition

$$\langle\psi(+\infty)|\psi(+\infty)\rangle = \sum_{f,f'} \langle f'|S^*_{f'i}S_{fi}|f\rangle = \sum_{f,f'} S^*_{f'i}S_{fi}\,\langle f'|f\rangle$$
$$= \sum_{f,f'} S^*_{f'i}S_{fi}\delta_{f'f} = \sum_f |S_{fi}|^2 = 1.$$

We can also write

$$\sum_f |S_{fi}|^2 = \sum_f S^*_{fi}S_{fi} = \sum_f S^\dagger_{if}S_{fi} = 1.$$

Thus we see that, if the probability of forming of bounds states is neglected. we can write

$$S^\dagger = S^{-1},$$

which means that the scattering operator S is unitary.

Scattering operator

Now we can use the normalisation condition

$$\langle\psi(+\infty)|\psi(+\infty)\rangle = \sum_{f,f'} \langle f'|S^*_{f'i} S_{fi}|f\rangle = \sum_{f,f'} S^*_{f'i} S_{fi} \langle f'|f\rangle$$

$$= \sum_{f,f'} S^*_{f'i} S_{fi} \delta_{f'f} = \sum_{f} |S_{fi}|^2 = 1.$$

We can also write

$$\sum_f |S_{fi}|^2 = \sum_f S^*_{fi} S_{fi} = \sum_f S^\dagger_{if} S_{fi} = 1.$$

Thus we see that, if the probability of forming of bounds states is neglected. we can write

$$S^\dagger = S^{-1},$$

which means that the scattering operator S is unitary.

Scattering operator

Let us note that the first term in the expansion of the evolution operator is equal to 1 $\Rightarrow$ It will not change the initial state. Therefore we write

$$S = 1 + T,$$

and then the matrix elements have the form

$$S_{fi} = \langle f|S|i\rangle = \langle f|i\rangle + \langle f|T|i\rangle = \delta_{fi} + T_{fi}.$$

Scattering operator

Let us note that the first term in the expansion of the evolution operator is equal to 1 $\Rightarrow$ It will not change the initial state. Therefore we write

$$S = 1 + T,$$

and then the matrix elements have the form

$$S_{fi} = \langle f|S|i\rangle = \langle f|i\rangle + \langle f|T|i\rangle = \delta_{fi} + T_{fi}.$$

The eigenstates of the Hamiltonian H_0 corresponding to different energies are orthogonal, thus $\langle f|i\rangle \neq 0$ only if $|f\rangle = |i\rangle$.

Let us note that the first term in the expansion of the evolution operator is equal to 1 $\Rightarrow$ It will not change the initial state. Therefore we write

$$S = 1 + T,$$

and then the matrix elements have the form

$$S_{fi} = \langle f|S|i\rangle = \langle f|i\rangle + \langle f|T|i\rangle = \delta_{fi} + T_{fi}.$$

The eigenstates of the Hamiltonian H_0 corresponding to different energies are orthogonal, thus $\langle f|i\rangle \neq 0$ only if $|f\rangle = |i\rangle$.

It is obvious that operator T can be expanded into the perturbative series

$$T = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 ... \int \mathrm{d}^4 x_n \, T \left[\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) ... \mathcal{H}_I(x_n) \right],$$

where we have put $\hbar = 1$. In QED

$$\mathcal{H}_I(x) = -e\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x).$$

Before we will be able to calculate matrix elements $\langle f|T|i\rangle$ the classical fields $\psi(x)$, $\bar{\psi}(x)$ and $A_{\mu}(x)$ in the definition of $\mathcal{H}_I(x)$ must be quantized.

Perturbative expansion of T

It is obvious that operator T can be expanded into the perturbative series

$$T = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 ... \int \mathrm{d}^4 x_n \, T \left[\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) ... \mathcal{H}_I(x_n) \right],$$

where we have put $\hbar = 1$. In QED

$$\mathcal{H}_I(x) = -e \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x).$$

Before we will be able to calculate matrix elements $\langle f | T | i \rangle$ the classical fields $\psi(x)$, $\bar{\psi}(x)$ and $A_\mu(x)$ in the definition of $\mathcal{H}_I(x)$ must be quantized.

Momentum space representation and quantization

In the course of Quantum Mechanics we have shown that the general solution of the free Dirac equation is a superposition of solutions with positive and negative energy:

$$\psi(x) = \sum_{\alpha} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left[c(\vec{k},\alpha) u^{(\alpha)}(k) e^{-ikx} + d(\vec{k},\alpha)^* v^{(\alpha)}(k) e^{ikx} \right],$$

where $k^0 = E = +\sqrt{\vec{k}^2 + m^2}$, the polarization index α takes 2 values, $\alpha = \pm\frac{1}{2}$, which usually are chosen as

- spin projection onto the Oz axis (canonical base) or
- spin projection on the particle momentum (helicity base),

and the integration measure

$$\frac{\mathrm{d}^3 k}{(2\pi)^3 2E}$$

is Lorentz invariant.

It can be easily shown that the general solution of the free Maxwell equation

$$\Box A^{\mu}(x) = 0, \quad \text{with the Lorentz condition} \quad \partial_{\mu} A^{\mu}(x) = 0$$

can be written as

$$A^{\mu}(x) = \sum_{\alpha=\pm 1} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left[a(\vec{k},\alpha)\varepsilon^{\mu}(k,\alpha)e^{-ikx} + a^{*}(\vec{k},\alpha)\varepsilon^{\mu}(k,\alpha)^{*}e^{ikx} \right]$$

where $k^0 = E = |\vec{k}|$ and polarization vectors $\varepsilon^{\mu}(k,\alpha)$ satisfy the following conditions

$$k_{\mu}\varepsilon^{\mu}(k,\alpha) = 0, \quad \varepsilon_{\mu}(k,\alpha')^{*}\varepsilon^{\mu}(k,\alpha) = -\delta_{\alpha'\alpha}.$$

Note that, although the photon is a spin 1 particle, $\alpha = \pm 1$, as polarization 0 is excluded for a massless particle.

It can be easily shown that the general solution of the free Maxwell equation

$$\Box A^\mu(x) = 0, \quad \text{with the Lorentz condition} \quad \partial_\mu A^\mu(x) = 0$$

can be written as

$$A^\mu(x) = \sum_{\alpha=\pm 1} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left[a(\vec{k},\alpha)\varepsilon^\mu(k,\alpha) e^{-ikx} + a^*(\vec{k},\alpha)\varepsilon^\mu(k,\alpha)^* e^{ikx} \right]$$

where $k^0 = E = |\vec{k}|$ and polarization vectors $\varepsilon^\mu(k,\alpha)$ satisfy the following conditions

$$k_\mu \varepsilon^\mu(k,\alpha) = 0, \quad \varepsilon_\mu(k,\alpha')^* \varepsilon^\mu(k,\alpha) = -\delta_{\alpha'\alpha}.$$

Note that, although the photon is a spin 1 particle, $\alpha = \pm 1$, as polarization 0 is excluded for a massless particle.

Quantization of the electromagnetic (EM) field $A^\mu(x)$ is not easy. The problem is the $U(1)$ gauge symmetry which is closely related to the fact that the photon is massless. We will leave this issue aside here and leave it to the course of QED.

At this point we only need to know that the EM field is quantized by imposing the following commutation relations on the operators $a(\vec{k},\alpha)$ and $a^\dagger(\vec{k},\alpha)$:

$$[a(\vec{k}',\alpha'), a^\dagger(\vec{k},\alpha)] = -g_{\alpha'\alpha}\delta^{(3)}(\vec{k}'-\vec{k})$$
$$[a(\vec{k}',\alpha'), a(\vec{k},\alpha)] = [a^\dagger(\vec{k}',\alpha'), a^\dagger(\vec{k},\alpha)] = 0,$$

where polarization indices $\alpha, \alpha' = 0, 1, 2, 3$.

Momentum space representation and quantization

Quantization of the electromagnetic (EM) field $A^\mu(x)$ is not easy. The problem is the $U(1)$ gauge symmetry which is closely related to the fact that the photon is massless. We will leave this issue aside here and leave it to the course of QED.

At this point we only need to know that the EM field is quantized by imposing the following commutation relations on the operators $a(\vec{k},\alpha)$ and $a^\dagger(\vec{k},\alpha)$:

$$[a(\vec{k}',\alpha'), a^\dagger(\vec{k},\alpha)] = -g_{\alpha'\alpha}\delta^{(3)}(\vec{k}' - \vec{k})$$
$$[a(\vec{k}',\alpha'), a(\vec{k},\alpha)] = [a^\dagger(\vec{k}',\alpha'), a^\dagger(\vec{k},\alpha)] = 0,$$

where polarization indices $\alpha, \alpha' = 0, 1, 2, 3$.

We will show that such bosonic quantization rules for the operators $a(\vec{k},\alpha)$ and $a^\dagger(\vec{k},\alpha)$ allow particle interpretation of them.

Momentum space representation and quantization

Quantization of the electromagnetic (EM) field $A^\mu(x)$ is not easy. The problem is the $U(1)$ gauge symmetry which is closely related to the fact that the photon is massless. We will leave this issue aside here and leave it to the course of QED.
At this point we only need to know that the EM field is quantized by imposing the following commutation relations on the operators $a(\vec{k}, \alpha)$ and $a^\dagger(\vec{k}, \alpha)$:

$$[a(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = -g_{\alpha'\alpha}\delta^{(3)}(\vec{k}' - \vec{k})$$
$$[a(\vec{k}', \alpha'), a(\vec{k}, \alpha)] = [a^\dagger(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = 0,$$

where polarization indices $\alpha, \alpha' = 0, 1, 2, 3$.
We will show that such bosonic quantization rules for the operators $a(\vec{k}, \alpha)$ and $a^\dagger(\vec{k}, \alpha)$ allow particle interpretation of them.

Particle interpretation of operators a and $a^\dagger$

Assume that operators a_α and $a^\dagger_\alpha$, where α stands for all possible quantum numbers which are necessary to fully describe the QM state, satisfy the following commutation rules

$$[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = [a^\dagger_\alpha, a^\dagger_\beta] = 0.$$

Let us define operator $N_\alpha = a^\dagger_\alpha a_\alpha$ and consider its eigenequation

$$N_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle .$$

As operator N_α is Hermitian, its eigenvalues n_α are real.

Let's calculate commutators $[N_\alpha, a_\beta]$ and $[N_\alpha, a^\dagger_\beta]$

$$[N_\alpha, a_\beta] = [a^\dagger_\alpha a_\alpha, a_\beta] = a^\dagger_\alpha [a_\alpha, a_\beta] + [a^\dagger_\alpha, a_\beta] a_\alpha = -\delta_{\alpha\beta} a_\alpha,$$
$$[N_\alpha, a^\dagger_\beta] = [a^\dagger_\alpha a_\alpha, a^\dagger_\beta] = a^\dagger_\alpha [a_\alpha, a^\dagger_\beta] + [a^\dagger_\alpha, a^\dagger_\beta] a_\alpha = \delta_{\alpha\beta} a^\dagger_\alpha.$$

Let's calculate

$$N_\alpha(a_\alpha |n_\alpha\rangle) = a_\alpha N_\alpha |n_\alpha\rangle - a_\alpha |n_\alpha\rangle = a_\alpha n_\alpha |n_\alpha\rangle - a_\alpha |n_\alpha\rangle$$
$$= n_\alpha a_\alpha |n_\alpha\rangle - a_\alpha |n_\alpha\rangle = (n_\alpha - 1)(a_\alpha |n_\alpha\rangle).$$

Particle interpretation of operators a and $a^\dagger$

Assume that operators a_α and $a_\alpha^\dagger$, where α stands for all possible quantum numbers which are necessary to fully describe the QM state, satisfy the following commutation rules

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0.$$

Let us define operator $N_\alpha = a_\alpha^\dagger a_\alpha$ and consider its eigenequation

$$N_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle .$$

As operator N_α is Hermitian, its eigenvalues n_α are real.
Let's calculate commutators $[N_\alpha, a_\beta]$ and $[N_\alpha, a_\beta^\dagger]$

$$[N_\alpha, a_\beta] = [a_\alpha^\dagger a_\alpha, a_\beta] = a_\alpha^\dagger [a_\alpha, a_\beta] + [a_\alpha^\dagger, a_\beta] a_\alpha = -\delta_{\alpha\beta} a_\alpha,$$
$$[N_\alpha, a_\beta^\dagger] = [a_\alpha^\dagger a_\alpha, a_\beta^\dagger] = a_\alpha^\dagger [a_\alpha, a_\beta^\dagger] + [a_\alpha^\dagger, a_\beta^\dagger] a_\alpha = \delta_{\alpha\beta} a_\alpha^\dagger.$$

Let's calculate

$$N_\alpha(a_\alpha |n_\alpha\rangle) = a_\alpha N_\alpha |n_\alpha\rangle - a_\alpha |n_\alpha\rangle = a_\alpha n_\alpha |n_\alpha\rangle - a_\alpha |n_\alpha\rangle$$
$$= n_\alpha a_\alpha |n_\alpha\rangle - a_\alpha |n_\alpha\rangle = (n_\alpha - 1)(a_\alpha |n_\alpha\rangle).$$

Particle interpretation of operators a and $a^\dagger$

Thus, we see that vector $a_\alpha |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 1$.

Similarly

$$N_\alpha(a_\alpha^\dagger |n_\alpha\rangle) = a_\alpha^\dagger N_\alpha |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle = a_\alpha^\dagger n_\alpha |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle$$
$$= n_\alpha a_\alpha^\dagger |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle = (n_\alpha + 1)(a_\alpha^\dagger |n_\alpha\rangle).$$

Thus, vector $a_\alpha^\dagger |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha + 1$.

Particle interpretation of operators a and $a^\dagger$

Thus, we see that vector $a_\alpha |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 1$.
Similarly

$$N_\alpha(a_\alpha^\dagger |n_\alpha\rangle) = a_\alpha^\dagger N_\alpha |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle = a_\alpha^\dagger n_\alpha |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle$$
$$= n_\alpha a_\alpha^\dagger |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle = (n_\alpha + 1)(a_\alpha^\dagger |n_\alpha\rangle).$$

Thus, vector $a_\alpha^\dagger |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha + 1$.
It is obvious that vector $a_\alpha^2 |n_\alpha\rangle = a_\alpha a_\alpha |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 2$ and so on.

Particle interpretation of operators a and $a^\dagger$

Thus, we see that vector $a_\alpha |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 1$.
Similarly

$$N_\alpha(a_\alpha^\dagger |n_\alpha\rangle) = a_\alpha^\dagger N_\alpha |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle = a_\alpha^\dagger n_\alpha |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle$$
$$= n_\alpha a_\alpha^\dagger |n_\alpha\rangle + a_\alpha^\dagger |n_\alpha\rangle = (n_\alpha + 1)(a_\alpha^\dagger |n_\alpha\rangle).$$

Thus, vector $a_\alpha^\dagger |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha + 1$.
It is obvious that vector $a_\alpha^2 |n_\alpha\rangle = a_\alpha a_\alpha |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 2$ and so on.
In this way, at some point we would reach negative values of n_α, which would exclude the particle interpretation of N_α, unless there exists the vacuum state, with no bosons, defined by

$$a_\alpha |0\rangle = 0, \qquad \text{for any } \alpha.$$

Particle interpretation of operators a and $a^\dagger$

Thus, we see that vector $a_\alpha \left|n_\alpha\right\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 1$.
Similarly

$$\begin{aligned} N_\alpha(a_\alpha^\dagger \left|n_\alpha\right\rangle) &= a_\alpha^\dagger N_\alpha \left|n_\alpha\right\rangle + a_\alpha^\dagger \left|n_\alpha\right\rangle = a_\alpha^\dagger n_\alpha \left|n_\alpha\right\rangle + a_\alpha^\dagger \left|n_\alpha\right\rangle \\ &= n_\alpha a_\alpha^\dagger \left|n_\alpha\right\rangle + a_\alpha^\dagger \left|n_\alpha\right\rangle = (n_\alpha + 1)(a_\alpha^\dagger \left|n_\alpha\right\rangle). \end{aligned}$$

Thus, vector $a_\alpha^\dagger \left|n_\alpha\right\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha + 1$.
It is obvious that vector $a_\alpha^2 \left|n_\alpha\right\rangle = a_\alpha a_\alpha \left|n_\alpha\right\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 2$ and so on.
In this way, at some point we would reach negative values of n_α, which would exclude the particle interpretation of N_α, unless there exists the vacuum state, with no bosons, defined by

$$a_\alpha \left|0\right\rangle = 0, \qquad \text{for any } \alpha.$$

Now, act with operator $a_\alpha^\dagger$ on both sides of this equation

$$a_\alpha^\dagger a_\alpha |0\rangle = 0\,|0\rangle = N_\alpha |0\rangle .$$

Thus, we see that the vacuum state corresponds to $n_\alpha = 0$.

Now, calculate

$$N_\alpha a_\alpha^\dagger |0\rangle = (0+1) a_\alpha^\dagger |0\rangle = a_\alpha^\dagger |0\rangle .$$

Now, act with operator $a^\dagger_\alpha$ on both sides of this equation

$$a^\dagger_\alpha a_\alpha |0\rangle = 0\,|0\rangle = N_\alpha |0\rangle\,.$$

Thus, we see that the vacuum state corresponds to $n_\alpha = 0$.
Now, calculate

$$N_\alpha a^\dagger_\alpha |0\rangle = (0+1)a^\dagger_\alpha |0\rangle = a^\dagger_\alpha |0\rangle\,.$$

Thus, we can define the states with $1, 2, \ldots$ particles in the QM state α by

$$|1_\alpha\rangle = a^\dagger_\alpha |0\rangle\,,$$
$$|2_\alpha\rangle = a^\dagger_\alpha |1_\alpha\rangle = (a^\dagger_\alpha)^2 |0\rangle\,,$$

for which $n_\alpha = 1, 2, \ldots$.

Particle interpretation of operators a and $a^\dagger$

Now, act with operator $a^\dagger_\alpha$ on both sides of this equation

$$a^\dagger_\alpha a_\alpha \left|0\right\rangle = 0 \left|0\right\rangle = N_\alpha \left|0\right\rangle .$$

Thus, we see that the vacuum state corresponds to $n_\alpha = 0$.
Now, calculate

$$N_\alpha a^\dagger_\alpha \left|0\right\rangle = (0+1) a^\dagger_\alpha \left|0\right\rangle = a^\dagger_\alpha \left|0\right\rangle .$$

Thus, we can define the states with $1, 2, \ldots$ particles in the QM state α by

$$\left|1_\alpha\right\rangle = a^\dagger_\alpha \left|0\right\rangle ,$$
$$\left|2_\alpha\right\rangle = a^\dagger_\alpha \left|1_\alpha\right\rangle = (a^\dagger_\alpha)^2 \left|0\right\rangle ,$$

for which $n_\alpha = 1, 2, \ldots$.

We see that N_α can be interpreted as the particle number operator, while a_α and $a^\dagger_\alpha$ as, respectively, annihilation and creation operators of a particle in the QM state α.

There can be arbitrarily many bosons in the same QM state α.

Now, act with operator $a_\alpha^\dagger$ on both sides of this equation

$$a_\alpha^\dagger a_\alpha |0\rangle = 0 |0\rangle = N_\alpha |0\rangle .$$

Thus, we see that the vacuum state corresponds to $n_\alpha = 0$.
Now, calculate

$$N_\alpha a_\alpha^\dagger |0\rangle = (0+1) a_\alpha^\dagger |0\rangle = a_\alpha^\dagger |0\rangle .$$

Thus, we can define the states with $1, 2, \ldots$ particles in the QM state α by

$$|1_\alpha\rangle = a_\alpha^\dagger |0\rangle ,$$
$$|2_\alpha\rangle = a_\alpha^\dagger |1_\alpha\rangle = (a_\alpha^\dagger)^2 |0\rangle ,$$

for which $n_\alpha = 1, 2, \ldots$.
We see that N_α can be interpreted as the particle number operator, while a_α and $a_\alpha^\dagger$ as, respectively, annihilation and creation operators of a particle in the QM state α.
There can be arbitrarily many bosons in the same QM state α.

Particle interpretation of a and $a^\dagger$

If we look again at the quantization rules for the EM field

$$[a(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = -g_{\alpha'\alpha}\delta^{(3)}(\vec{k}' - \vec{k})$$
$$[a(\vec{k}', \alpha'), a(\vec{k}, \alpha)] = [a^\dagger(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = 0,$$

where polarization indices $\alpha, \alpha' = 0, 1, 2, 3$, we will immediately see that there is a problem with particle interpretation, as operators $a(\vec{k}, \alpha)$ and $a^\dagger(\vec{k}, \alpha)$ change there roles for $\alpha = 0$, i.e. $a(\vec{k}, 0)$ should be considered as the creation and $a^\dagger(\vec{k}, 0)$ as the annihilation operator of a scalar photon with momentum $\vec{k}$.

The problem was solved by restricting the Hilbert space to the physical subspace in which all the states $|\psi_{\rm phys.}\rangle$ satisfy the Lorentz condition in a weak form

$$\partial^\mu A_\mu^{(+)}(x)\,|\psi_{\rm phys.}\rangle = 0$$

Particle interpretation of a and $a^\dagger$

If we look again at the quantization rules for the EM field

$$[a(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = -g_{\alpha'\alpha}\delta^{(3)}(\vec{k}' - \vec{k})$$

$$[a(\vec{k}', \alpha'), a(\vec{k}, \alpha)] = [a^\dagger(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = 0,$$

where polarization indices $\alpha, \alpha' = 0, 1, 2, 3$, we will immediately see that there is a problem with particle interpretation, as operators $a(\vec{k}, \alpha)$ and $a^\dagger(\vec{k}, \alpha)$ change there roles for $\alpha = 0$, i.e. $a(\vec{k}, 0)$ should be considered as the creation and $a^\dagger(\vec{k}, 0)$ as the annihilation operator of a scalar photon with momentum $\vec{k}$.
The problem was solved by restricting the Hilbert space to the physical subspace in which all the states $|\psi_{\rm phys.}\rangle$ satisfy the Lorentz condition in a weak form

$$\partial^\mu A_\mu^{(+)}(x)\, |\psi_{\rm phys.}\rangle = 0$$

It can be shown that due to this condition contributions from photons of the scalar $(\alpha = 0)$ and longitudinal $(\alpha = 3)$ polarizations to any physical observable cancel each other.

Particle interpretation of a and $a^\dagger$

If we look again at the quantization rules for the EM field

$$[a(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = -g_{\alpha'\alpha}\delta^{(3)}(\vec{k}' - \vec{k})$$
$$[a(\vec{k}', \alpha'), a(\vec{k}, \alpha)] = [a^\dagger(\vec{k}', \alpha'), a^\dagger(\vec{k}, \alpha)] = 0,$$

where polarization indices $\alpha, \alpha' = 0, 1, 2, 3$, we will immediately see that there is a problem with particle interpretation, as operators $a(\vec{k}, \alpha)$ and $a^\dagger(\vec{k}, \alpha)$ change there roles for $\alpha = 0$, i.e. $a(\vec{k}, 0)$ should be considered as the creation and $a^\dagger(\vec{k}, 0)$ as the annihilation operator of a scalar photon with momentum $\vec{k}$. The problem was solved by restricting the Hilbert space to the physical subspace in which all the states $|\psi_{\mathrm{phys.}}\rangle$ satisfy the Lorentz condition in a weak form

$$\partial^\mu A_\mu^{(+)}(x)\, |\psi_{\mathrm{phys.}}\rangle = 0$$

It can be shown that due to this condition contributions from photons of the scalar $(\alpha = 0)$ and longitudinal $(\alpha = 3)$ polarizations to any physical observable cancel each other.

Particle interpretation of a and $a^\dagger$

Therefore, the free EM field can be written as a sum of the *positive* $(+)$ and *negative* $(-)$ frequency parts

$$A_\mu(x) = A_\mu^{(+)}(x) + A_\mu^{(-)}(x),$$

where

$$A_\mu^{(+)}(x) = \sum_{\alpha=\pm1} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} a(\vec{k},\alpha)\varepsilon^\mu(k,\alpha) e^{-ikx},$$

$$A_\mu^{(-)}(x) = \sum_{\alpha=\pm1} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} a^\dagger(\vec{k},\alpha)\varepsilon^\mu(k,\alpha)^* e^{ikx},$$

where $k^0 = E = |\vec{k}|$ and we sum only over transverse polarizations $(\alpha = \pm 1)$.

We see, that operator $A_\mu^{(+)}(x)$ annihilates a photon of any momentum $\vec{k}$ and polarization α at the space time point x.

Similarly, operator $A_\mu^{(-)}(x)$ creates a photon of any momentum $\vec{k}$ and polarization α at this point.

Quantization of a fermion fields

Assume that operators c_α and $c_\alpha^\dagger$, where α stands for all possible quantum numbers which are necessary to fully describe the QM state, satisfy the following anti commutation rules

$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}, \qquad \{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0.$$

Let's define operator $N_\alpha = c_\alpha^\dagger c_\alpha$ and consider its eigenequation

$$N_\alpha \left| n_\alpha \right\rangle = n_\alpha \left| n_\alpha \right\rangle .$$

Quantization of a fermion fields

Assume that operators c_α and $c_\alpha^\dagger$, where α stands for all possible quantum numbers which are necessary to fully describe the QM state, satisfy the following anti commutation rules

$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}, \qquad \{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0.$$

Let's define operator $N_\alpha = c_\alpha^\dagger c_\alpha$ and consider its eigenequation

$$N_\alpha \left| n_\alpha \right\rangle = n_\alpha \left| n_\alpha \right\rangle .$$

We again need commutators $[N_\alpha, c_\beta]$ and $[N_\alpha, c_\beta^\dagger]$.

Quantization of a fermion fields

Assume that operators c_α and $c_\alpha^\dagger$, where α stands for all possible quantum numbers which are necessary to fully describe the QM state, satisfy the following anti commutation rules

$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}, \qquad \{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0.$$

Let's define operator $N_\alpha = c_\alpha^\dagger c_\alpha$ and consider its eigenequation

$$N_\alpha \left| n_\alpha \right\rangle = n_\alpha \left| n_\alpha \right\rangle .$$

We again need commutators $[N_\alpha, c_\beta]$ and $[N_\alpha, c_\beta^\dagger]$.

Note that

$$[AB, C] = ABC - CAB + ACB - ACB = A\{B, C\} - \{A, C\}B.$$

Thus

$$[N_\alpha, c_\beta] = [c_\alpha^\dagger c_\alpha, c_\beta] = c_\alpha^\dagger\{c_\alpha, c_\beta\} - \{c_\alpha^\dagger, c_\beta\}c_\alpha = -\delta_{\alpha\beta}c_\alpha,$$
$$[N_\alpha, c_\beta^\dagger] = [c_\alpha^\dagger c_\alpha, c_\beta^\dagger] = c_\alpha^\dagger\{c_\alpha, c_\beta^\dagger\} - \{c_\alpha^\dagger, c_\beta^\dagger\}c_\alpha = \delta_{\alpha\beta}c_\alpha^\dagger.$$

Quantization of a fermion fields

Assume that operators c_α and $c_\alpha^\dagger$, where α stands for all possible quantum numbers which are necessary to fully describe the QM state, satisfy the following anti commutation rules

$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}, \qquad \{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0.$$

Let's define operator $N_\alpha = c_\alpha^\dagger c_\alpha$ and consider its eigenequation

$$N_\alpha \left|n_\alpha\right\rangle = n_\alpha \left|n_\alpha\right\rangle .$$

We again need commutators $[N_\alpha, c_\beta]$ and $[N_\alpha, c_\beta^\dagger]$.
Note that

$$[AB, C] = ABC - CAB + ACB - ACB = A\{B, C\} - \{A, C\}B.$$

Thus

$$[N_\alpha, c_\beta] = [c_\alpha^\dagger c_\alpha, c_\beta] = c_\alpha^\dagger\{c_\alpha, c_\beta\} - \{c_\alpha^\dagger, c_\beta\}c_\alpha = -\delta_{\alpha\beta}c_\alpha,$$
$$[N_\alpha, c_\beta^\dagger] = [c_\alpha^\dagger c_\alpha, c_\beta^\dagger] = c_\alpha^\dagger\{c_\alpha, c_\beta^\dagger\} - \{c_\alpha^\dagger, c_\beta^\dagger\}c_\alpha = \delta_{\alpha\beta}c_\alpha^\dagger.$$

Calculate

$$N_\alpha(c_\alpha |n_\alpha\rangle) = c_\alpha N_\alpha |n_\alpha\rangle - c_\alpha |n_\alpha\rangle = c_\alpha n_\alpha |n_\alpha\rangle - c_\alpha |n_\alpha\rangle$$
$$= n_\alpha c_\alpha |n_\alpha\rangle - c_\alpha |n_\alpha\rangle = (n_\alpha - 1)(c_\alpha |n_\alpha\rangle).$$

We see that vector $c_\alpha |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 1$. It means that operator c_α annihilates a fermion in the QM state α.

Similarly

$$N_\alpha(c_\alpha^\dagger |n_\alpha\rangle) = c_\alpha^\dagger N_\alpha |n_\alpha\rangle + c_\alpha^\dagger |n_\alpha\rangle = c_\alpha^\dagger n_\alpha |n_\alpha\rangle + c_\alpha^\dagger |n_\alpha\rangle$$
$$= n_\alpha c_\alpha^\dagger |n_\alpha\rangle + c_\alpha^\dagger |n_\alpha\rangle = (n_\alpha + 1)(c_\alpha^\dagger |n_\alpha\rangle).$$

Thus, vector $c_\alpha^\dagger |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha + 1$. It means that operator $c_\alpha^\dagger$ creates a fermion in the QM state α.

Calculate

$$N_\alpha(c_\alpha |n_\alpha\rangle) = c_\alpha N_\alpha |n_\alpha\rangle - c_\alpha |n_\alpha\rangle = c_\alpha n_\alpha |n_\alpha\rangle - c_\alpha |n_\alpha\rangle$$
$$= n_\alpha c_\alpha |n_\alpha\rangle - c_\alpha |n_\alpha\rangle = (n_\alpha - 1)(c_\alpha |n_\alpha\rangle).$$

We see that vector $c_\alpha |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha - 1$. It means that operator c_α annihilates a fermion in the QM state α.

Similarly

$$N_\alpha(c_\alpha^\dagger |n_\alpha\rangle) = c_\alpha^\dagger N_\alpha |n_\alpha\rangle + c_\alpha^\dagger |n_\alpha\rangle = c_\alpha^\dagger n_\alpha |n_\alpha\rangle + c_\alpha^\dagger |n_\alpha\rangle$$
$$= n_\alpha c_\alpha^\dagger |n_\alpha\rangle + c_\alpha^\dagger |n_\alpha\rangle = (n_\alpha + 1)(c_\alpha^\dagger |n_\alpha\rangle).$$

Thus, vector $c_\alpha^\dagger |n_\alpha\rangle$ is the eigenvector of N_α to eigenvalue $n_\alpha + 1$. It means that operator $c_\alpha^\dagger$ creates a fermion in the QM state α.

Quantization of a fermion fields

Now, let us note that

$$N_\alpha^2 = c_\alpha^\dagger c_\alpha c_\alpha^\dagger c_\alpha = c_\alpha^\dagger(-c_\alpha^\dagger c_\alpha + 1)c_\alpha = -c_\alpha^\dagger c_\alpha^\dagger c_\alpha c_\alpha + c_\alpha^\dagger c_\alpha = N_\alpha,$$

where, due to the anti commutation rules, we put
$c_\alpha c_\alpha = c_\alpha^\dagger c_\alpha^\dagger = 0$.
We have obtained the operator equation

$$N_\alpha^2 = N_\alpha \quad \Rightarrow \quad (N_\alpha - 1)N_\alpha = 0$$

and hence for the eigenvalues we obtain

$$(N_\alpha - 1)N_\alpha |n_\alpha\rangle = (n_\alpha - 1)n_\alpha |n_\alpha\rangle = 0,$$

i.e., $n_\alpha = 1$ or $n_\alpha = 0$, which means that there can be just one or no fermions in any QM state α.
This explains the Pauli exclusion principle that was introduced as a postulate of QM.

Now, let us note that

$$N_\alpha^2 = c_\alpha^\dagger c_\alpha c_\alpha^\dagger c_\alpha = c_\alpha^\dagger(-c_\alpha^\dagger c_\alpha + 1)c_\alpha = -c_\alpha^\dagger c_\alpha^\dagger c_\alpha c_\alpha + c_\alpha^\dagger c_\alpha = N_\alpha,$$

where, due to the anti commutation rules, we put $c_\alpha c_\alpha = c_\alpha^\dagger c_\alpha^\dagger = 0.$

We have obtained the operator equation

$$N_\alpha^2 = N_\alpha \quad \Rightarrow \quad (N_\alpha - 1)N_\alpha = 0$$

and hence for the eigenvalues we obtain

$$(N_\alpha - 1)N_\alpha \left|n_\alpha\right\rangle = (n_\alpha - 1)n_\alpha \left|n_\alpha\right\rangle = 0,$$

i.e., $n_\alpha = 1$ or $n_\alpha = 0$, which means that there can be just one or no fermions in any QM state α.

This explains the Pauli exclusion principle that was introduced as a postulate of QM.

Momentum space representation and quantization

In quantum field theory (QFT) we impose the following anti commutation relations on operators $c(\vec{k},\alpha)$ and $d(\vec{k},\alpha)$:

$$\{c(\vec{k}',\alpha'),c^\dagger(\vec{k},\alpha)\} = \{d(\vec{k}',\alpha'),d^\dagger(\vec{k},\alpha)\} = \delta_{\alpha'\alpha}\delta^{(3)}(\vec{k}'-\vec{k})$$

$$\{c(\vec{k}',\alpha'),c(\vec{k},\alpha)\} = \{d(\vec{k}',\alpha'),d(\vec{k},\alpha)\} = \{c(\vec{k}',\alpha'),d^\dagger(\vec{k},\alpha)\} = 0$$

and write fermion fields $\psi(x)$ and $\bar{\psi}(x)$ in the following form

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x), \qquad \bar{\psi}(x) = \bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x),$$

where

$$\psi^{(+)}(x) = \sum_\alpha \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} c(\vec{k},\alpha)u^{(\alpha)}(k)e^{-ikx},$$

$$\psi^{(-)}(x) = \sum_\alpha \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} d^\dagger(\vec{k},\alpha)v^{(\alpha)}(k)e^{ikx},$$

$$\bar{\psi}^{(+)}(x) = \sum_\alpha \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} d(\vec{k},\alpha)\bar{v}^{(\alpha)}(k)e^{-ikx},$$

$$\bar{\psi}^{(-)}(x) = \sum_{\alpha} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} c^{\dagger}(\vec{k}, \alpha) \bar{u}^{(\alpha)}(k) e^{ikx},$$

with $u^{(\alpha)}(k)$ and $v^{(\alpha)}(k)$ being spinors of a particle and antiparticle, respectively, and $k^0 = E = +\sqrt{\vec{k}^2 + m^2}$.

Thus, we see that field operator

- $\psi^{(+)}(x)$ annihilates a fermion,
- $\psi^{(-)}(x)$ creates an antifermion,
- $\bar{\psi}^{(+)}(x)$ annihilates an antifermion,
- $\bar{\psi}^{(-)}(x)$ creates a fermion

of any momentum $\vec{k}$ and polarization state α.

$$\bar{\psi}^{(-)}(x) = \sum_{\alpha} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} c^{\dagger}(\vec{k}, \alpha) \bar{u}^{(\alpha)}(k) e^{ikx},$$

with $u^{(\alpha)}(k)$ and $v^{(\alpha)}(k)$ being spinors of a particle and antiparticle, respectively, and $k^0 = E = +\sqrt{\vec{k}^2 + m^2}$.
Thus, we see that field operator

- $\psi^{(+)}(x)$ annihilates a fermion,
- $\psi^{(-)}(x)$ creates an antifermion,
- $\bar{\psi}^{(+)}(x)$ annihilates an antifermion,
- $\bar{\psi}^{(-)}(x)$ creates a fermion

of any momentum $\vec{k}$ and polarization state α.

Quantization of fields

Imposing the bosonic quantization rules for operators $a(\vec{k}, \alpha)$ and $a^\dagger(\vec{k}, \alpha)$ and fermionic quantization rules for operators $c(\vec{k}, \alpha)$, $c^\dagger(\vec{k}, \alpha)$, $d(\vec{k}, \alpha)$ and $d^\dagger(\vec{k}, \alpha)$ is sometimes referred to as the *second quantization*.

This is not correct, as those rules directly follow from the quantization rules which we impose on fields and their conjugate momenta.

Imposing the bosonic quantization rules for operators $a(\vec{k},\alpha)$ and $a^\dagger(\vec{k},\alpha)$ and fermionic quantization rules for operators $c(\vec{k},\alpha)$, $c^\dagger(\vec{k},\alpha)$, $d(\vec{k},\alpha)$ and $d^\dagger(\vec{k},\alpha)$ is sometimes referred to as the *second quantization*.

This is not correct, as those rules directly follow from the quantization rules which we impose on fields and their conjugate momenta.

The bosonic quantization rules for operators $a(\vec{k},\alpha)$ and $a^\dagger(\vec{k},\alpha)$ are just a simple generalization of the quantization rules of QM imposed on coordinates and conjugate momenta, with the only difference that the field describes a physical system with infinite (uncountable) number of degrees of freedom.

Imposing the bosonic quantization rules for operators $a(\vec{k},\alpha)$ and $a^\dagger(\vec{k},\alpha)$ and fermionic quantization rules for operators $c(\vec{k},\alpha)$, $c^\dagger(\vec{k},\alpha)$, $d(\vec{k},\alpha)$ and $d^\dagger(\vec{k},\alpha)$ is sometimes referred to as the *second quantization*.

This is not correct, as those rules directly follow from the quantization rules which we impose on fields and their conjugate momenta.

The bosonic quantization rules for operators $a(\vec{k},\alpha)$ and $a^\dagger(\vec{k},\alpha)$ are just a simple generalization of the quantization rules of QM imposed on coordinates and conjugate momenta, with the only difference that the field describes a physical system with infinite (uncountable) number of degrees of freedom.

For example, for the real scalar field $\varphi(x)$, which is described by the Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \frac{1}{2}\varphi(x)^2$$

we define the conjugate momentum by

$$\pi(t,\vec{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}(t,\vec{x})}$$

and qntize it by imposing the following simultaneous commutation relations ($\hbar = c = 1$)

$$[\varphi(t,\vec{x}'),\pi(t,\vec{x})] = i\delta^{(3)}(\vec{x}'-\vec{x}),$$
$$[\varphi(t,\vec{x}'),\varphi(t,\vec{x})] = [\pi(t,\vec{x}'),\pi(t,\vec{x})] = 0,$$

which exactly correspond to the quantization rules of QM

$$[x_i,p_j] = i\hbar\delta_{ij}, \quad [x_i,x_j] = [p_i,p_j] = 0.$$

Perturbative expansion of T_{fi}

The derivation of the momentum representation of T is quite involved, as the momentum representation of all the fields must be inserted in every interaction Hamiltonian density $\mathcal{H}_I(x)$ in the expansion formula of the scattering operator T.

In QED, which is by far the simplest realistic QFT, we have

$$\mathcal{H}_I(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) = -e\left(\bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x)\right)$$
$$\gamma^\mu\left(\psi^{(+)}(x) + \psi^{(-)}(x)\right)\left(A_\mu^{(+)}(x) + A_\mu^{(-)}(x)\right).$$

Thus, there are 8 terms for each appearance of the Hamiltonian $\mathcal{H}_I(x)$ in the perturbative series of T_{fi}, i.e. 8^n terms in the n-th term of the series

$$\langle f|T|i\rangle = \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int \mathrm{d}^4x_1 \ldots \int \mathrm{d}^4x_n \,\langle f|T\left[:\mathcal{H}_I(x_1): \ldots :\mathcal{H}_I(x_n):\right]|i\rangle$$
$$\equiv \sum_{n=1}^{\infty}\left\langle f|T^{(n)}|i\right\rangle,$$

Perturbative expansion of T_{fi}

The derivation of the momentum representation of T is quite involved, as the momentum representation of all the fields must be inserted in every interaction Hamiltonian density $\mathcal{H}_I(x)$ in the expansion formula of the scattering operator T.
In QED, which is by far the simplest realistic QFT, we have

$$\mathcal{H}_I(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) = -e\left(\bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x)\right)$$
$$\gamma^\mu\left(\psi^{(+)}(x) + \psi^{(-)}(x)\right)\left(A_\mu^{(+)}(x) + A_\mu^{(-)}(x)\right).$$

Thus, there are 8 terms for each appearance of the Hamiltonian $\mathcal{H}_I(x)$ in the perturbative series of T_{fi}, i.e. 8^n terms in the n-th term of the series

$$\langle f|T|i\rangle = \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int \mathrm{d}^4x_1 \ldots \int \mathrm{d}^4x_n \langle f|T\left[:\mathcal{H}_I(x_1): \ldots :\mathcal{H}_I(x_n):\right]|i\rangle$$
$$\equiv \sum_{n=1}^{\infty}\left\langle f|T^{(n)}|i\right\rangle,$$

where we have assumed that each term of $<f|T^{(n)}|i>$ is brought to the so called *normal order*, where the annihilation operators stand to the right and creation operators to the left, in order to avoid infinities resulting from action of creation operators on the vacuum $|0\rangle$, because of the plethora of operators we choose just those which will annihilate the initial state to the vacuum and then create the final state of a considered reaction.

To clarify this issue let's find the normal order of

$$:\mathcal{H}_I:= -e :\left(\bar{\psi}^{(+)} + \bar{\psi}^{(-)}\right)\left(\not{A}^{(+)} + \not{A}^{(-)}\right)\left(\psi^{(+)} + \psi^{(-)}\right):$$

$$= -e :\left(\bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)}\psi_b^{(+)} + \bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)}\psi_b^{(-)} + \bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\psi_b^{(+)} + \bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\psi_b^{(-)}\right.$$
$$\left. +\bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(+)} + \bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(-)} + \bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(+)} + \bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(-)}\right):$$

$$= -e\left(\bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)}\psi_b^{(+)} - \psi_b^{(-)}\bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)} + \bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\psi_b^{(+)} - \psi_b^{(-)}\bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\right.$$
$$\left. +\bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(+)} + \bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(-)} + \bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(+)} + \bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(-)}\right),$$

where we have assumed that all operators (anti-)commute, where we have assumed that each term of $<f|T^{(n)}|i>$ is brought to the so called *normal order*, where the annihilation operators stand to the right and creation operators to the left, in order to avoid infinities resulting from action of creation operators on the vacuum $|0\rangle$, because of the plethora of operators we choose just those which will annihilate the initial state to the vacuum and then create the final state of a considered reaction.

To clarify this issue let's find the normal order of

$$:\mathcal{H}_I:= -e:\left(\bar{\psi}^{(+)}+\bar{\psi}^{(-)}\right)\left(\not{A}^{(+)}+\not{A}^{(-)}\right)\left(\psi^{(+)}+\psi^{(-)}\right):$$

$$= -e:\left(\bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)}\psi_b^{(+)}+\bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)}\psi_b^{(-)}+\bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\psi_b^{(+)}+\bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\psi_b^{(-)}\right.$$
$$\left.+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(+)}+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(-)}+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(+)}+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(-)}\right):$$
$$= -e\left(\bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)}\psi_b^{(+)}-\psi_b^{(-)}\bar{\psi}_a^{(+)}\not{A}_{ab}^{(+)}+\bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\psi_b^{(+)}-\psi_b^{(-)}\bar{\psi}_a^{(+)}\not{A}_{ab}^{(-)}\right.$$
$$\left.+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(+)}+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(+)}\psi_b^{(-)}+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(+)}+\bar{\psi}_a^{(-)}\not{A}_{ab}^{(-)}\psi_b^{(-)}\right),$$

where we have assumed that all operators (anti-)commute, i.e. the right hand sides of the corresponding (anti-)commutation relations are discarded.

All the remaining operators in $<f|T^{(n)}|i>$ must be joined to form the Feynman propagators, which are Green's functions defined in the following way

$$iS_F(x-x') \equiv \left\langle 0|T(\psi(x)\bar{\psi}(x'))|0\right\rangle,$$

with

$$T(\psi(x)\bar{\psi}(x')) = \theta(t-t')\psi(x)\bar{\psi}(x') - \theta(t'-t)\bar{\psi}(x')\psi(x)$$

for the fermionic field and

$$iD_F^{\mu\nu}(x-x') \equiv \langle 0|T\left(A^\mu(x)A^\nu(x')\right)|0\rangle,$$

with

$$T\left(A^\mu(x)A^\nu(x')\right) = \theta(t-t')A^\mu(x)A^\nu(x') + \theta(t'-t)A^\nu(x')A^\mu(x),$$

for the photon field.

i.e. the right hand sides of the corresponding (anti-)commutation relations are discarded.
All the remaining operators in $<f|T^{(n)}|i>$ must be joined to form the Feynman propagators, which are Green's functions defined in the following way

$$iS_F(x-x') \equiv \left\langle 0 \right| T(\psi(x)\bar{\psi}(x'))\left| 0 \right\rangle,$$

with

$$T(\psi(x)\bar{\psi}(x')) = \theta(t-t')\psi(x)\bar{\psi}(x') - \theta(t'-t)\bar{\psi}(x')\psi(x)$$

for the fermionic field and

$$iD_F^{\mu\nu}(x-x') \equiv \langle 0| T\left(A^\mu(x)A^\nu(x')\right)|0\rangle,$$

with

$$T\left(A^\mu(x)A^\nu(x')\right) = \theta(t-t')A^\mu(x)A^\nu(x') + \theta(t'-t)A^\nu(x')A^\mu(x),$$

for the photon field.

Perturbative expansion of T_{fi}

How exactly this done is strictly described by the Wick's theorem which can be proved in QFT.

Here we only need to know the Fourier transforms of the Feynman propagator of a fermion and a photon which read as follows

$$iS_F(x-x') = \int \frac{d^4k}{(2\pi)^4} i \frac{\not{k}+m}{k^2-m^2+i\varepsilon} e^{-ik(x-x')},$$

$$iD_F^{\mu\nu}(x-x') = \int \frac{d^4k}{(2\pi)^4} i \frac{-g^{\mu\nu}}{k^2+i\varepsilon} e^{-ik(x-x')},$$

where the photon propagator is defined in the Feynman gauge. Due to the $U(1)$ gauge symmetry of QED the gauge choice is arbitrary and the Feynman gauge is the simplest and most convenient choice for most applications.

How exactly this done is strictly described by the Wick's theorem which can be proved in QFT.
Here we only need to know the Fourier transforms of the Feynman propagator of a fermion and a photon which read as follows

$$iS_F(x-x') = \int \frac{d^4k}{(2\pi)^4} i \frac{\not{k}+m}{k^2-m^2+i\varepsilon} e^{-ik(x-x')},$$
$$iD_F^{\mu\nu}(x-x') = \int \frac{d^4k}{(2\pi)^4} i \frac{-g^{\mu\nu}}{k^2+i\varepsilon} e^{-ik(x-x')},$$

where the photon propagator is defined in the Feynman gauge. Due to the $U(1)$ gauge symmetry of QED the gauge choice is arbitrary and the Feynman gauge is the simplest and most convenient choice for most applications.

Perturbative expansion of T_{fi}

It can be also shown that the space integral of the exponential factors $e^{\pm ikx}$ in the Fourier transforms of the Feynman propagators and field operators always result in the following factorization of the matrix element T_{fi}

$$T_{fi} = (2\pi)^4 \delta^{(4)}(\sum_i p_i - \sum_f p_f) M_{fi}.$$

The factor

$$(2\pi)^4 \delta^{(4)}(\sum_i p_i - \sum_f p_f)$$

is usually associated with the Lorentz invariant phase space element $\mathrm{d}Lips$, as we did in the derivation of the relativistic cross section formula.

Perturbative expansion of T_{fi}

It can be also shown that the space integral of the exponential factors $e^{\pm ikx}$ in the Fourier transforms of the Feynman propagators and field operators always result in the following factorization of the matrix element T_{fi}

$$T_{fi} = (2\pi)^4 \delta^{(4)}(\sum_i p_i - \sum_f p_f) M_{fi}.$$

The factor

$$(2\pi)^4 \delta^{(4)}(\sum_i p_i - \sum_f p_f)$$

is usually associated with the Lorentz invariant phase space element d$Lips$, as we did in the derivation of the relativistic cross section formula.

As an example, let us calculate the cross section of

$e^+e^- \to l^+l^-$, where $l = \mu, \tau$.

Perturbative expansion of T_{fi}

It can be also shown that the space integral of the exponential factors $e^{\pm ikx}$ in the Fourier transforms of the Feynman propagators and field operators always result in the following factorization of the matrix element T_{fi}

$$T_{fi} = (2\pi)^4 \delta^{(4)}(\sum_i p_i - \sum_f p_f) M_{fi}.$$

The factor

$$(2\pi)^4 \delta^{(4)}(\sum_i p_i - \sum_f p_f)$$

is usually associated with the Lorentz invariant phase space element $\mathrm{d}Lips$, as we did in the derivation of the relativistic cross section formula.

As an example, let us calculate the cross section of

$e^+e^- \to l^+l^-$, where $l = \mu, \tau$.