Dirac Equation

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Schrödinger Equation

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$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\,\vec{\nabla}^2\psi(\vec{r},t) + V(\vec{r},t)\psi(\vec{r},t)$$

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The squared module of the wave function $|\psi(\vec{r},t)|^2$ is interpreted as probability density of finding the particle in the spatial volume element $\mathrm{d}^3r = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ around the point $\vec{r}$ at the time t.

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Probability Current

We can define the probability current for a spinless non relativistic particle as the vector

$$\vec{S}(\vec{r},t) = -\frac{i\hbar}{2m}\left[\psi^*\vec{\nabla}\psi - \left(\vec{\nabla}\psi^*\right)\psi\right]$$

which, together with the probability density $|\psi(\vec{r},t)|^2$, satisfies the following continuity equation

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Relativistic Wave Equation

It seems that the simplest way to obtain a relativistic wave equation would be the following substitution

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \vec{p} \to -i\hbar\vec{\nabla}$$

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$$E^2 - \vec{p}^2 c^2 = m^2 c^4,$$

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Let us remind that the relationship $E^2 - \vec{p}^2 c^2 = m^2 c^4$ is Lorentz invariant, as it can be derived by calculating the inner (dot) product of the energy-momentum four vector

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

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$$E = \gamma m c^2, \qquad \vec{p} = \gamma m \vec{v},$$

where the Lorentz factor γ is given by

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which after multiplying its both sides by c^2 gives our starting relationship $E^2 - \vec{p}^2c^2 = m^2c^4$.

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where we have introduced symbol $\Box$ to denote the d'Alembert operator.
Moreover, let us denote $\mu^2 \equiv \frac{m^2c^2}{\hbar^2}$, then the equation

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Exercise. Show that ∂_μ is a covariant and ∂^μ is a contravariant four vector.

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Let us now calculate the four divergence of the current $j^{\mu}(x)$

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Klein-Gordon Wave Equation

Let us now calculate the four divergence of the current $j^\mu(x)$

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Klein-Gordon Wave Equation

We would like to interpret the zeroth component $j^0(x)$ of the current $j^\mu(x)$ as the probability density $\rho(x)$ of finding a particle in the spatial volume element d^3x at time t.

Unfortunately,

$$\rho(x) = i\left[\varphi^*(x)\partial^0\varphi(x) - \partial^0\left(\varphi^*(x)\right)\varphi(x)\right]$$

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If, in spite of that, we assumed that $|\varphi(x)|^2$ is the probability density of finding a particle in the spatial volume element d^3x at time t, then the probability normalization integral

$$\int |\varphi(t, \vec{x})|^2 \, \mathrm{d}^3x = 1$$

would not be time conserved.

Another problem is related to the fact that there is the second derivative with respect to time in the Klein-Gordon equation, while we would like the relativistic wave equation to have the form analogous to the Schrödinger equation, i.e.

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If we take the square root of the relationship $E^2 - \vec{p}^2 c^2 = m^2 c^4$, then we obtain the relativistic formula for the particle energy

$$E = \pm\sqrt{\vec{p}^2 c^2 + m^2 c^4}.$$

We want the particle energy to be positive, thus we choose

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Klein-Gordon Wave Equation

To see this let us Fourier transform the wave function

$$\psi(t,\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3 k e^{i\vec{k}\cdot\vec{x}} \tilde{\psi}(t,\vec{k}).$$

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Exercise. Show that

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Now let us calculate

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Let us calculate square of the Hamilton operator

$$
\begin{aligned}
H^2 &= (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m) \\
&= \alpha_i \alpha_j p_i p_j + \alpha_i p_i \beta m + \beta m \alpha_j p_j + \beta^2 m^2 \\
&= \frac{1}{2}\alpha_i \alpha_j p_i p_j + \frac{1}{2}\alpha_j \alpha_i p_j p_i + \alpha_i \beta m p_i + \beta \alpha_i m p_i + \beta^2 m^2 \\
&= \frac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + (\alpha_i \beta + \beta \alpha_i) m p_i + \beta^2 m^2,
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And now check the commutation properties of γ^μ's: $\gamma^0 \equiv \beta$, $\gamma^i \equiv \beta\alpha_i$, $i = 1, 2, 3$. Remember that

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}\mathbb{I}, \qquad \alpha_i\beta + \beta\alpha_i = 0, \qquad \beta^2 = \mathbb{I}.$$

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Let us summarize our results

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and recall the form of the metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We see that

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Dirac Equation

The equations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I}, \qquad \mu, \nu = 0, 1, 2, 3,$$

together with the Hermiticity properties $\gamma^{0\,\dagger} = \gamma^0, \quad \gamma^{i\,\dagger} = -\gamma^i$ can be considered as definition of the Dirac matrices γ^μ, $\mu = 0, 1, 2, 3.$

The Dirac matrices can be chosen in the following way

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Exercise. Show that γ^μ matrices in the Dirac representation satisfy the following anticommutation relationships

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Having found one representation of the Dirac matrices we can obtain any other representation by means of a unitary transformation.

$$\gamma^\mu \;\rightarrow\; \tilde{\gamma}^\mu = U^\dagger \gamma^\mu U, \quad \text{where} \quad UU^\dagger = U^\dagger U = \mathbb{I}.$$

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Let us note that, if the new γ^μ's were defined by the similarity transform

$$\gamma^\mu \;\rightarrow\; \tilde{\gamma}^\mu = S^{-1}\gamma^\mu S,$$

then the new matrices would not satisfy desired Hermiticity properties.

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Dirac Matrices

The other commonly used representation of matrices γ^μ is the Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

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To this end let us write down explicitly the matrix indices in the Dirac equation

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and, after performing the multiplication, we get

$$\left(-\gamma^{\nu}_{ca}\gamma^{\mu}_{ab}\partial_{\nu}\partial_{\mu} - im\gamma^{\nu}_{cb}\partial_{\nu} + im\gamma^{\mu}_{cb}\partial_{\mu} - m^2\delta_{cb}\right)\psi_b(x) = 0.$$

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Dirac Equation

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$$E = \pm\sqrt{\vec{p}^2 + m^2}.$$

Existence of the negative energy solutions caused some anxiety in the beginning, but then it occurred that they just represent antiparticles of positive energy $E = \sqrt{\vec{p}^2 + m^2}$ which propagate opposite to the time flow.

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Conjugate Dirac Equation

Let us take the Hermitian conjugate of the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0,$$

then we obtain

$$\psi^\dagger(x)\left(-i\,\gamma^{\mu\,\dagger}\,\overleftarrow{\partial}_\mu - m\right) = 0,$$

where an arrow above the derivative means that it acts to the left and not to the right, as usual.

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Dirac Current

Multiplying the Dirac equation from the left by $\bar{\psi}(x)$ we get

$$\bar{\psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0.$$

Now let us add both sides of this equation to the equation

$$i\partial_{\mu}\bar{\psi}(x)\gamma^{\mu}\psi(x) + m\bar{\psi}(x)\psi(x) = 0,$$

Dirac Current

Multiplying the Dirac equation from the left by $\bar{\psi}(x)$ we get

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If we divide both sides of it by i we get

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Dirac Current

Let us calculate the zeroth component of the Dirac current

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi$$

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\begin{aligned}
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Thus, we see that the zeroth component of the Dirac current can be interpreted as the probability density ρ of finding a particle in the spatial volume element d^3x at time t.

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