Bell's inequalities

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Introduction

By 1935, it was already recognized that the predictions of quantum mechanics (QM) are probabilistic. In their famous paper of 1935 Albert Einstein, Boris Podolsky and Nathan Rosen presented a scenario that, in their view, indicated that quantum particles, like electrons and photons, must carry physical properties or attributes not included in QM, and the uncertainties in predictions of QM were due to ignorance of these properties, later termed *hidden variables*.

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Bell's inequalities were introduced by John Stewart Bell in a 1964 paper titled *On the Einstein Podolsky Rosen Paradox*. They show that QM is incompatible with local hidden-variable theories.

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Angular momentum operator

The angular momentum operator $\vec{J}$ can be defined as the Hermitian operator which satisfies the following commutation rules

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k, \quad \text{for} \quad i, j = 1, 2, 3.$$

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Exercise. Show that

$$\left[\vec{J}^2, J_i\right] = 0 \quad \text{i} \quad \vec{J}^{2\dagger} = \vec{J}^2, \quad \text{for} \quad i = 1,2,3.$$

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Using the above definition it can be shown that

$$\begin{aligned} \vec{J}^2 \, |jm\rangle &= j(j+1)\hbar^2 \, |jm\rangle \,, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \\ J_3 \, |jm\rangle &= m\hbar \, |jm\rangle \,, \qquad m = -j, -j+1, \ldots, j-1, j. \end{aligned}$$

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Matrix representations of J_i

Using the Hermiticity and commutation properties of the angular momentum operators J_i, $i = 1, 2, 3$, it can also be shown that matrix elements of operators $J_\pm = J_1 \pm iJ_2$ have the following form

$$\begin{aligned} \langle jm+1|J_+|jm\rangle &= [j(j+1) - m(m+1)]^{\frac{1}{2}}\,\hbar \\ \langle jm-1|J_-|jm\rangle &= [j(j+1) - m(m-1)]^{\frac{1}{2}}\,\hbar, \end{aligned}$$

or, assuming an arbitrary complex phase equal to 1, we can write

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where we have used the eigenequation of operator J_3 and orthonormality of the angular momentum eigenvectors $|jm\rangle$.

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Using the relationship

$$J_+ \left| jm \right\rangle = [j(j+1) - m(m+1)]^{\frac{1}{2}} \hbar \left| jm+1 \right\rangle$$

we find the matrix representing operator J_+

$$\begin{aligned} J_+ &= \begin{pmatrix} \left\langle \frac{1}{2}, \frac{1}{2} \middle| J_+ \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, \frac{1}{2} \middle| J_+ \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\ \left\langle \frac{1}{2}, -\frac{1}{2} \middle| J_+ \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2} \middle| J_+ \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{pmatrix} \\ &= \hbar \left(\begin{matrix} \left(\frac{3}{4} - \frac{3}{4}\right)^{\frac{1}{2}} \left\langle \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}, \frac{3}{2} \right\rangle & \left(\frac{3}{4} + \frac{1}{4}\right)^{\frac{1}{2}} \left\langle \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}, \frac{1}{2} \right\rangle \\ & \end{matrix} \right. \end{aligned}$$

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we find the matrix representing operator J_-

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Matrix representations of J_i

Now we can easily find matrices representing operators J_1 and J_2.

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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Exercise. Find matrices representing operators J_i, $i = 1, 2, 3$ for $j = 1$, $j = 3/2$ and $j = 2$.

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Exercise. Find matrices representing operators J_i, $i = 1, 2, 3$ for $j = 1$, $j = 3/2$ and $j = 2$.

Consider a spin $\frac{1}{2}$ particle. The spin operator has the form

$$\vec{S} = \frac{1}{2}\vec{\sigma},$$

where we have assumed $\hbar = 1$.

The spin components can be measured with the Stern–Gerlach device by means of projection on the inhomogeneous magnetic field.

Spin correlations in a singlet state

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Let $\hat{a}$ denote the unit vector in the direction of the inhomogeneous magnetic field. Instead of spin projection onto it, i.e., $\hat{a} \cdot \vec{S}$ it is more convenient to use the observable

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If we choose $\hat{a} = \hat{e}_3$ then we will obtain

$$\not{e}_3 = 2\hat{e}_3 \cdot \vec{S} = \sigma_3,$$

with the eigenvectors $\not{e}_3 \left|\pm\right\rangle = \pm \left|\pm\right\rangle$ of the following form

$$\left|+\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \left|-\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Obviously

$$\left\langle+\right| = (1 \;\; 0)\,, \qquad \left\langle-\right| = (0 \;\; 1)\,.$$

Spin correlations in a singlet state

Now, recall that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and calculate

$$\not{a} = \hat{a} \cdot \vec{\sigma} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}.$$

Let us find the eigenvalues of $\not{a}$. To this end we have to solve the equation

$$\begin{vmatrix} a_3 - \lambda & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 - \lambda \end{vmatrix} = \lambda^2 - a_3^2 - a_1^2 - a_2^2 = 0.$$

As $\hat{a}$ is the unit vector, we get $\lambda = \pm 1$.

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Spin correlations in a singlet state

Thus, the eigenequation of $\not{a}$ has the form

$$\not{a}\,|\hat{a}\pm\rangle = \pm\,|\hat{a}\pm\rangle\,.$$

Vector $\hat{a}$ can be obtained from vector $\hat{e}_3$ by a rotation of angle $\vec{\theta} = \theta\hat{\theta}$, with $\hat{\theta}$ being a unit vector parallel to $\hat{e}_3 \times \hat{a}$, which determines the direction of the rotation axis. Hence, we have

$$|\hat{a}\pm\rangle = e^{-i\vec{\theta}\cdot\vec{S}}\,|\pm\rangle\,,$$

where

$$e^{-i\vec{\theta}\cdot\vec{S}} S_3 e^{i\vec{\theta}\cdot\vec{S}} = \hat{a}\cdot\vec{S}.$$

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Let us calculate matrix elements of $\not{a}$ in the spin eigenvector basis

$$\langle +|\not{a}|+\rangle = (1\ \ 0)\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ \ 0)\begin{pmatrix} a_3 \\ a_1 + ia_2 \end{pmatrix}$$
$$= a_3,$$
$$\langle -|\not{a}|-\rangle = (0\ \ 1)\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0\ \ 1)\begin{pmatrix} a_1 - ia_2 \\ -a_3 \end{pmatrix}$$
$$= -a_3,$$
$$\langle +|\not{a}|-\rangle = (1\ \ 0)\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1\ \ 0)\begin{pmatrix} a_1 - ia_2 \\ -a_3 \end{pmatrix}$$
$$= a_1 - ia_2,$$
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$$= a_1 + ia_2.$$

Spin correlations in a singlet state

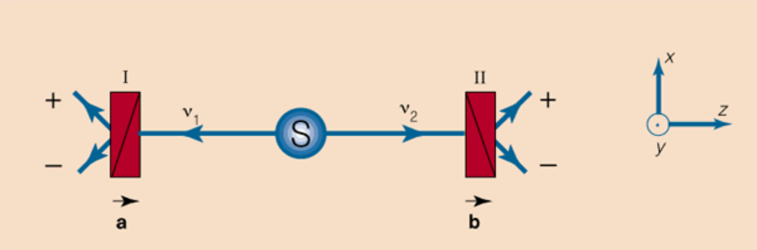

Figure: Simultaneous spin measurements on particle pairs $(1) + (2)$. S is the particle source, and $\vec{a}$ and $\vec{b}$ are field directions of the Stern–Gerlach magnets.

Now we consider the combination of two different spin-$\frac{1}{2}$ systems. A system of basis vectors is

$$|(1)+\rangle \otimes |(2)+\rangle\,, \quad |(1)-\rangle \otimes |(2)-\rangle\,,$$
$$|(1)+\rangle \otimes |(2)-\rangle\,, \quad |(1)-\rangle \otimes |(2)+\rangle\,,$$

where (1) and (2) refer to the first and second particle, respectively, and $+$ and $-$ specifies the z component of the spin.
We can also use the Stern–Gerlach device that measures the spin component of the first particle along $\hat{a}$ and the spin component of the second particle along $\hat{b}$.

Spin correlations in a singlet state

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Then we take the following basis system

$$|\hat{a}+\rangle \otimes \left|\hat{b}+\right\rangle, \quad |\hat{a}-\rangle \otimes \left|\hat{b}-\right\rangle, \quad |\hat{a}+\rangle \otimes \left|\hat{b}-\right\rangle, \quad |\hat{a}-\rangle \otimes \left|\hat{b}+\right\rangle,$$

with two arbitrarily chosen vectors unit vectors $\hat{a}$ and $\hat{b}$.

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with two arbitrarily chosen vectors unit vectors $\hat{a}$ and $\hat{b}$.
Consider the following two observables

$$\not{a} \otimes \mathbb{I} = 2\hat{a} \cdot \vec{S} \otimes \mathbb{I} \quad \text{and} \quad \mathbb{I} \otimes \not{b} = \mathbb{I} \otimes 2\hat{b} \cdot \vec{S},$$

which are 2 $\times$ spin component of particle (1) along $\hat{a}$ and 2 $\times$ spin component of particle (2) along $\hat{b}$, respectively.

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Spin correlations in a singlet state

They act on our basis vectors $|\hat{a}\alpha\rangle \otimes \left|\hat{b}\beta\right\rangle$, $\alpha, \beta = \pm 1$, in the following way

$$\not{a} \otimes \mathbb{I} \, |\hat{a}\alpha\rangle \otimes \left|\hat{b}\beta\right\rangle = \not{a} \, |\hat{a}\alpha\rangle \otimes \mathbb{I} \left|\hat{b}\beta\right\rangle = \alpha \, |\hat{a}\alpha\rangle \otimes \left|\hat{b}\beta\right\rangle$$
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The two observables commute and our basis vectors are simultaneously eigenvectors of both of them with eigenvalues $+1$ or -1. Indeed, let us calculate

$$[\not{a} \otimes \mathbb{I}, \mathbb{I} \otimes \not{b}] = (\not{a} \otimes \mathbb{I})(\mathbb{I} \otimes \not{b}) - (\mathbb{I} \otimes \not{b})(\not{a} \otimes \mathbb{I})$$
$$= \not{a}\mathbb{I} \otimes \mathbb{I}\not{b} - \mathbb{I}\not{a} \otimes \not{b}\mathbb{I} = \not{a} \otimes \not{b} - \not{a} \otimes \not{b} = 0.$$

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Thus $\not{a} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \not{b}$ can be measured simultaneously.

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Spin correlations in a singlet state

The measurement of $\not{a} \otimes \mathbb{I}$ in the basis states

$$|\hat{a}+\rangle \otimes \left|\hat{b}+\right\rangle, \quad |\hat{a}-\rangle \otimes \left|\hat{b}-\right\rangle, \quad |\hat{a}+\rangle \otimes \left|\hat{b}-\right\rangle, \quad |\hat{a}-\rangle \otimes \left|\hat{b}+\right\rangle,$$

will always yield $+1, -1, +1, -1$, and the simultaneous measurement of $\mathbb{I} \otimes \not{b}$ will yield $+1, -1, -1, +1$.

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For such simultaneous measurement we can in addition define a *spin correlation observable*, which is by definition the product of the values obtained in a single measurement of both $\not{a} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \not{b}$.

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The simultaneous spin measurements on two-particle systems with two Stern–Gerlach devices are possible only if the two particles of each pair are spatially separated and each particle moves along a certain fixed axis, as shown in the Figure below.

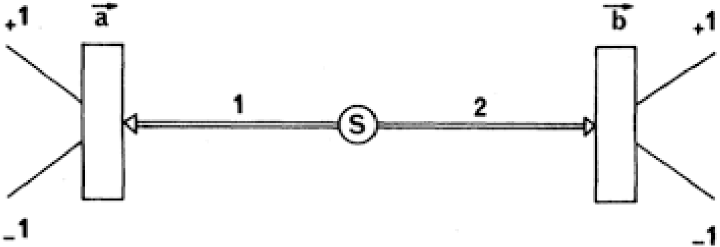

Figure: Simultaneous spin measurements on particle pairs $(1) + (2)$.

A particle source emits pairs of particles, one pair at a time, such that particle (1) is always emitted to the left, and particle (2) is always emitted to the right.

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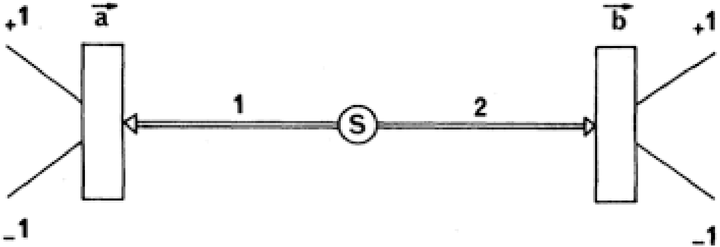

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Then a Stern–Gerlach device with inhomegenous magnetic field along some direction $\hat{a}$, perpendicular to the beam, may be applied to the left beam and another Stern–Gerlach device with field direction $\hat{b}$ may be applied to the right beam. Each device has two counters, one at a position $+1$ and and the other at a position -1. Since the particles (1) and (2) are emitted pairwise by the source, the two particles of a single pair pass the two Stern–Gerlach magnets an arrive at two of the four counters almost simultaneously.

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Spin correlations in a singlet state

This kind of measurement is repeated N times, with $N \gg 1$, and the following numbers are recorded:

- the number N_{++} of simultaneous clicks of $+1$ counter on the left and $+1$ counter on the right,
- the number N_{+-} of simultaneous clicks of $+1$ counter on the left and -1 counter on the right,

the numbers N_{-+} and N_{--} are defined similarly.

The measured *average values* for the observables $\not{a} \otimes \mathbb{I}$, $\mathbb{I} \otimes \not{b}$ and $\not{a} \otimes \not{b}$, which we denote, respectively, by $E_1(\hat{a})$, $E_2(\hat{b})$ and $E(\hat{a}, \hat{b})$ are the following:

$$E_1(\hat{a}) = \frac{1}{N}(N_{++} + N_{+-} - N_{-+} - N_{--}),$$

$$E_2(\hat{b}) = \frac{1}{N}(N_{++} - N_{+-} + N_{-+} - N_{--}),$$

$$E(\hat{a}, \hat{b}) = \frac{1}{N}(N_{++} - N_{+-} - N_{-+} + N_{--}),$$

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Spin correlations in a singlet state

According to quantum mechanics, these measured average values should coincide with the *expectation values* of corresponding observables in the common spin state of the particle pairs emitted by the source.

The combination of two spin-$\frac{1}{2}$ systems may lead to the total spin value

$$s = \left|\frac{1}{2} - \frac{1}{2}\right|, ..., \frac{1}{2} + \frac{1}{2},$$

i.e., $s = 0$ or $s = 1$.

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We will assume here that the particle pairs emitted by the source have total spin 0, and are therefore in the asymmetric singlet state

$$|\phi\rangle = \frac{1}{\sqrt{2}}\big(|(1)+\rangle \otimes |(2)-\rangle - |(1)-\rangle \otimes |(2)+\rangle\big)$$
$$\equiv \frac{1}{\sqrt{2}}\big(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\big).$$

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$$\equiv \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right).$$

Spin correlations in a singlet state

A source emitting particle pairs in the spin state $|\phi\rangle$ might contain, e.g., a large number of unstable compounds of two particles (1) and (2) at rest, and therefore after the decay, due to momentum conservation, particles (1) and (2) move always in opposite directions. However, while moving away, they are still in the common spin state $|\phi\rangle$.

We can now calculate the corresponding QM expectation values.

$$\begin{aligned}\langle\phi|\not{a}\otimes\mathbb{I}|\phi\rangle &= \frac{1}{2}(\langle+|\otimes\langle-|-\langle-|\otimes\langle+|)\not{a}\otimes\mathbb{I}(|+\rangle\otimes|-\rangle-|-\rangle\otimes|+\rangle) \\ &= \frac{1}{2}(\langle+|\otimes\langle-|-\langle-|\otimes\langle+|)(\not{a}\,|+\rangle\otimes|-\rangle-\not{a}\,|-\rangle\otimes|+\rangle) \\ &= \frac{1}{2}(\langle+|\not{a}|+\rangle\langle-|-\rangle-\langle+|\not{a}|-\rangle\langle-|+\rangle-\langle-|\not{a}|+\rangle\langle+|-\rangle \\ &\qquad +\langle-|\not{a}|-\rangle\langle+|+\rangle) = \frac{1}{2}(a_3-a_3)=0,\end{aligned}$$

where we have used $\langle+|\not{a}|+\rangle = a_3$ and $\langle-|\not{a}|-\rangle = -a_3$.

Spin correlations in a singlet state

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Spin correlations in a singlet state

Using $\langle +|\not{a}|+\rangle = a_3$, $\langle -|\not{a}|-\rangle = -a_3$, $\langle +|\not{a}|-\rangle = a_1 - ia_2$ and $\langle -|\not{a}|+\rangle = a_1 + ia_2$, and analogously for $\not{b}$, we will get

$$\langle\phi|\mathbb{I}\otimes\not{b}|\phi\rangle = \frac{1}{2}(\langle +|\otimes\langle -| - \langle -|\otimes\langle +|)\mathbb{I}\otimes\not{b}(|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle)$$
$$= \frac{1}{2}(\langle +|\otimes\langle -| - \langle -|\otimes\langle +|)(|+\rangle\otimes\not{b}\,|-\rangle - |-\rangle\otimes\not{b}\,|+\rangle)$$
$$= \frac{1}{2}(\langle +|+\rangle\langle -|\not{b}|-\rangle - \langle +|-\rangle\langle -|\not{b}|+\rangle - \langle -|+\rangle\langle +|\not{b}|-\rangle$$
$$+ \langle -|-\rangle\langle +|\not{b}|+\rangle) = \frac{1}{2}(-b_3 + b_3) = 0.$$

Similarly

$$\langle\phi|\not{a}\otimes\not{b}|\phi\rangle = \frac{1}{2}(\langle +|\not{a}|+\rangle\langle -|\not{b}|-\rangle - \langle +|\not{a}|-\rangle\langle -|\not{b}|+\rangle - \langle -|\not{a}|+\rangle\langle +|\not{b}|-\rangle$$
$$+ \langle -|\not{a}|-\rangle\langle +|\not{b}|+\rangle) = \frac{1}{2}\left(a_3(-b_3) - (a_1 - ia_2)(b_1 + ib_2)\right.$$
$$\left.-(a_1 + ia_2)(b_1 - ib_2) + (-a_3)b_3\right) = -\hat{a}\cdot\hat{b}.$$

Spin correlations in a singlet state

The QM predictions for the expectation values $\langle\phi|\not{a}\otimes\mathbb{I}|\phi\rangle$, $\langle\phi|\mathbb{I}\otimes\not{b}|\phi\rangle$ and $\langle\phi|\not{a}\otimes\not{b}|\phi\rangle$ hold obviously for a large number of single particle pair spin measurements.

Compare the QM prediction for

$$\langle\phi|\not{a}\otimes\mathbb{I}|\phi\rangle = 0$$

with the measured average value

$$E_1(\hat{a}) = \frac{1}{N}(N_{++} + N_{+-} - N_{-+} - N_{--}).$$

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We see that the QM prediction implies

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which means that the number of cases in which the spin of particle (1) is found to be parallel and atiparallel to $\hat{a}$ are equal for any choice of $\hat{a}$. This result is a consequence of the rotational invariance of the spin state $|\phi\rangle$.

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Spin correlations in a singlet state

The same conclusion can be derived for particle (2) if we compare the QM prediction for $\langle\phi|\mathbb{I}\otimes \not{b}|\phi\rangle$ with the measured average value of

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At the first glance comparison of the QM prediction for the spin correlation $\langle\phi|\not{a}\otimes \not{b}|\phi\rangle$ with the measured average value of

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Spin correlations in a singlet state

Then, remembering that $N = N_{++} + N_{+-} + N_{-+} + N_{--}$ we will get the following condition

$$N_{++} - N_{+-} - N_{-+} + N_{--} = -N_{++} - N_{+-} - N_{-+} - N_{--}$$

and, since both N_{++} and N_{--} are positive, we see that

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According to the QM prediction

$$E(\hat{a}, \hat{a}) = -1$$

spin components of two particles (1) and (2) along a fixed direction $\hat{a}$ are always opposite to each other.
Instead of directly measuring $\not{a} \otimes \mathbb{I}$ on particle (1) itself, we can equally well determine its spin component along $\hat{a}$ by measuring $\mathbb{I} \otimes \not{a}$ on particle (2) and multiplying the result by -1.

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Therefore, it seems quite *natural* to imagine that a single particle (1) does not *get* a definite spin component along $\hat{a}$ during a measurement of $\not{a} \otimes \mathbb{I}$, but rather *has* a definite value of it, either $+1$ or -1, prior to and independent of any measurement.

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Assume that prior to and independent of any measurement every single particle (1) possesses a definite value $v(\hat{a})$, of either $+1$ or -1, for the components of its spin, at least along all possible directions $\hat{a}$ orthogonal to the beam.

These values are just *uncovered*, rather than *produced*, if the actual spin measurement is performed. They may be visualized as hidden labels, either $+1$ or -1, attached to every single particle (1) for every possible direction $\hat{a}$. The same argument applies obviously to all particles (2).

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Without this assumption it seems quite difficult to understand the perfect anticorrelation $\langle\phi|\not{a}\otimes\not{a}|\phi\rangle = -1$ for simultaneous measurements of $\not{a}\otimes\mathbb{I}$ on particle (1) and $\mathbb{I}\otimes\not{a}$ on particle (2).

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For if the value of $\not{a}\otimes\mathbb{I}$ was really undetermined until it is actually measured on particle (1), it would appear impossible for particle (2), which may be very far away, to *get informed* about this value, in order to be able to *choose* just the opposite value.

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Consider a very large number N of particle pairs in the spin singlet state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$$

and four arbitrarily chosen directions $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ in the plane orthogonal to the two beams of particles produced by the source. Denote by $v_i(\hat{a})$ and $v_i(\hat{d})$ the *hidden* predetermined values of the spin components along $\hat{a}$ and $\hat{d}$ of particle (1) in the i-th pair, and by $w_i(\hat{b})$ and $w_i(\hat{c})$ the *hidden* predetermined values of the spin components along $\hat{b}$ and $\hat{c}$ of particle (2) in the same pair.

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Then, if the spin component of particle (1) along $\hat{a}$ and the spin component of particle (2) along $\hat{b}$ are measured simultaneously for all N pairs the average spin correlation value becomes

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But we could have chosen other directions, e.g., $\hat{d}$ in the left and $\hat{c}$ in the right beam, for the orientation of the two Stern–Gerlach devices.

If such experiment had been performed instead with *the same N* particle pairs, it would have *uncovered* the spin components $v_i(\hat{d})$ and $w_i(\hat{c})$ and the observed spin correlation average would have been

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It seems reasonable to assume that an experimental determination of $E(\hat{d}, \hat{c})$ *would have produced the same result* if it had been performed on any of the four sets of N particle pairs.

Therefore $E(\hat{a}, \hat{b})$, $E(\hat{d}, \hat{c})$, $E(\hat{a}, \hat{c})$ and $E(\hat{d}, \hat{b})$ may be considered as quantities characteristic of the N particle pairs set and their preparation, and are not dependent on which particular experiment is actually performed.

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Although we know the QM predictions for all all of them, i.e., $-\hat{a} \cdot \hat{b}$, $-\hat{d} \cdot \hat{c}$, $-\hat{a} \cdot \hat{c}$ and $-\hat{d} \cdot \hat{b}$, respectively, we will not yet assume this, but we will use the *experimental* results given of the previous slide.

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To this end let us first show that

$$v_i(\hat{a})(w_i(\hat{b}) + w_i(\hat{c})) + v_i(\hat{d})(w_i(\hat{b}) - w_i(\hat{c})) = \pm 2,$$

$i = 1, 2, ..., N$.

Proof. As w_i is either $+1$ or -1, the first bracket is either $+2, 0$ or -2.

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$$v_i(\hat{a})(w_i(\hat{b}) + w_i(\hat{c})) + v_i(\hat{d})(w_i(\hat{b}) - w_i(\hat{c})) = \pm 2,$$

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If we now sum all the above equations over $i = 1, 2, ..., N$ we will obtain the inequality

$$-2N \leq \sum_{i=1}^{N} \left[v_i(\hat{a})(w_i(\hat{b}) + w_i(\hat{c})) + v_i(\hat{d})(w_i(\hat{b}) - w_i(\hat{c})) \right] \leq 2N,$$

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and dividing this by N we obtain

$$\left|\frac{1}{N}\sum_{i=1}^{N} v_i(\hat{a})w_i(\hat{b}) + \frac{1}{N}\sum_{i=1}^{N} v_i(\hat{a})w_i(\hat{c}) + \frac{1}{N}\sum_{i=1}^{N} v_i(\hat{d})w_i(\hat{b}) - \frac{1}{N}\sum_{i=1}^{N} v_i(\hat{d})w_i(\hat{c})\right| \leq 2.$$

Thus, we obtain the inequality

$$\left|E(\hat{a},\hat{b}) + E(\hat{a},\hat{c}) + E(\hat{d},\hat{b}) - E(\hat{d},\hat{c})\right| \leq 2,$$

which is the most famous and experimentally most useful of a series of similar inequalities known as *Bell's inequalities.*

Let us now check if the QM prediction

$$E(\hat{a}, \hat{b}) = \langle \phi | \not{a} \otimes \not{b} | \phi \rangle = -\hat{a} \cdot \hat{b}$$

satisfies the above Bell's inequality.

$$\begin{aligned} & | - \hat{a} \cdot \hat{b} - \hat{a} \cdot \hat{c} - \hat{d} \cdot \hat{b} + \hat{d} \cdot \hat{c} | = | \hat{a} \cdot \hat{b} + \hat{a} \cdot \hat{c} + \hat{d} \cdot \hat{b} - \hat{d} \cdot \hat{c} | \\ & = | \hat{a} \cdot (\hat{b} + \hat{c}) + \hat{d} \cdot (\hat{b} - \hat{c}) | \leq |\hat{a}||\hat{b} + \hat{c}| + |\hat{d}||\hat{b} - \hat{c}| \\ & = |\hat{b} + \hat{c}| + |\hat{b} - \hat{c}| = \sqrt{(\hat{b} + \hat{c})^2} + \sqrt{(\hat{b} - \hat{c})^2} \\ & = \sqrt{2 + 2\cos\theta} + \sqrt{2 - 2\cos\theta}, \end{aligned}$$

with θ being the angle between $\hat{b}$ and $\hat{c}$, $\hat{b} \cdot \hat{c} = \cos\theta$, $\theta \in [0, \pi]$.

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Denote the expression on the right hand side of our inequality by

$$f(\theta) = \sqrt{2 + 2\cos\theta} + \sqrt{2 - 2\cos\theta} = 2\sqrt{\frac{1+\cos\theta}{2}} + 2\sqrt{\frac{1-\cos\theta}{2}},$$

which for $\theta \in [0, \pi]$ can be written as

$$f(\theta) = 2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}.$$

Let us find the maximum of $f(\theta)$.

$$f'(\theta) = -\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = 0 \quad \Leftrightarrow \quad \frac{\theta}{2} = \frac{\pi}{4}.$$

Thus $f(\theta) = 0$ for $\theta = \frac{\pi}{2}$. Calculate

$$f''(\theta) = -\frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{2}\sin\frac{\theta}{2}\Big|_{\theta=\frac{\pi}{2}} = -2\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2} = -2\sqrt{2} < 0.$$

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Thus, $f(\theta)$ has the maximum at $\theta = \frac{\pi}{2}$ equal to

$$f(\theta) = \sqrt{2 + 2\cos\frac{\pi}{2}} + \sqrt{2 - 2\cos\frac{\pi}{2}} = 2\sqrt{2}$$

and the QM prediction for our inequality is the following

$$\left| E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{c}) + E(\hat{d}, \hat{b}) - E(\hat{d}, \hat{c}) \right| \leq 2\sqrt{2}.$$

The Bell inequality becomes equality, i.e., it is maximally violated by the QM prediction, if

1. $\hat{a}$ and $\hat{b} + \hat{c}$, and $\hat{d}$ and $\hat{b} - \hat{c}$ are parallel,
2. $\hat{a}$ and $\hat{b} + \hat{c}$, and $\hat{d}$ and $\hat{b} - \hat{c}$ are antiparallel.

These configurations are depicted in the Figure below.

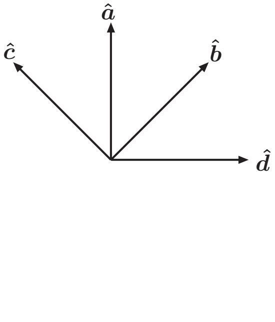

Figure: Magnetic field configurations of the Stern–Gerlach devices for which Bell's inequality is maximally violated.

The equality

$$\left| E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{c}) + E(\hat{d}, \hat{b}) - E(\hat{d}, \hat{c}) \right| = 2\sqrt{2}$$

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This is not as easy in practice as one might imagine, however.
First of all it is difficult to prepare pairs of spin-$\frac{1}{2}$ particles in a spin singlet state $|\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$.

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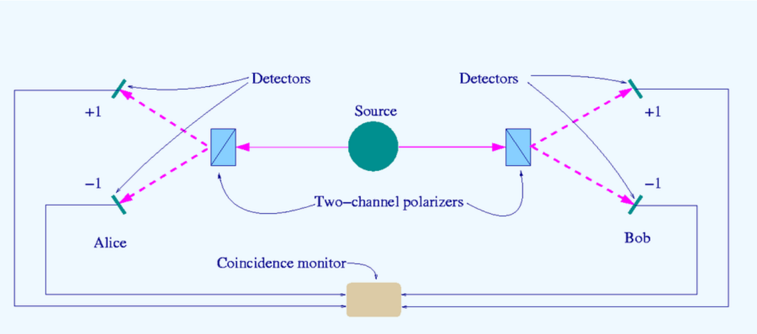
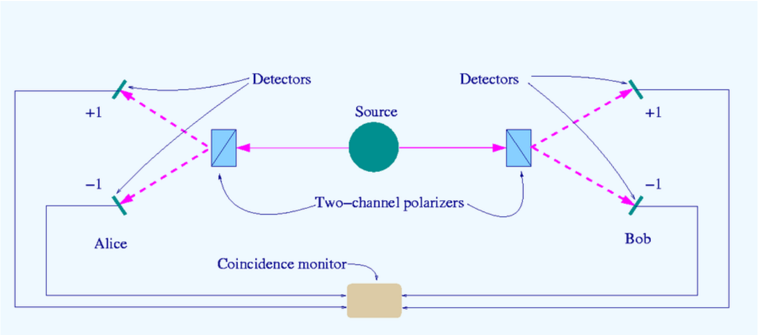


Figure: Scheme of a photon analyzer for tests of Bell's inequalities.

In this case, the simultaneous spin measurements discussed till now are replaced by simultaneous measurements of the transverse linear polarizations of the two emitted photons along arbitrarily chosen directions $\hat{a}$ and $\hat{b}$.



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In this case, the simultaneous spin measurements discussed till now are replaced by simultaneous measurements of the transverse linear polarizations of the two emitted photons along arbitrarily chosen directions $\hat{a}$ and $\hat{b}$.

For the photon pairs emitted in cascade transitions, these polarizations are correlated in much the same way as the spin components of spin-$\frac{1}{2}$ particle pairs in the considered singlet state $|\phi\rangle$.

However, the QM prediction for $E(\hat{a}, \hat{b})$ is different and therefore the configurations of directions $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ for which Bell's inequality

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Moreover, as $v_i(\hat{d})$ and $w_i(\hat{c})$ simply *do not exist* the equation

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The hypothesis that two spin-$\frac{1}{2}$ particles *have* definite spin components $v_i(\hat{a})$ and $w_i(\hat{b})$ prior to the measurement seems natural only from the classical point of view, in which we visualize the particle pair as consisting of two separate particles.

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However, the QM state vector

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|(1)+\rangle \otimes |(2)-\rangle - |(1)-\rangle \otimes |(2)+\rangle)$$

does not describe a state with separate single-particle properties.
Such states would be described by any one of the basis vectors

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The QM state vector $|\phi\rangle$ describes a new entity, an *indivisible whole*, a single object whose constituent particles (1) and (2) are not definable until a measurement is made that prepares the direct product states $|(1)+\rangle \otimes |(2)+\rangle$, $|(1)-\rangle \otimes |(2)-\rangle$, $|(1)+\rangle \otimes |(2)-\rangle$ and $|(1)-\rangle \otimes |(2)+\rangle$, or their mixtures.

Being a state with the total spin 0, $|\phi\rangle$ does not have single-particle properties. And as a *pure* QM state it cannot be subdivided.

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In classical physics, the building blocks of a composite system are its constituents. In QM the building blocks are subspaces of the Hilbert space of physical states. They can be any kind of subspaces, not only the one-dimensional subspaces spanned by any of the direct products listed above. The state $|\phi\rangle$ is one of the multitude of arbitrary linear combinations of them.

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The *counterintuitive* features of QM, as those we have just discussed, were difficult to accept for many physicists, who grew up in the classical tradition.

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Being a state with the total spin 0, $|\phi\rangle$ does not have single-particle properties. And as a *pure* QM state it cannot be subdivided.

In classical physics, the building blocks of a composite system are its constituents. In QM the building blocks are subspaces of the Hilbert space of physical states. They can be any kind of subspaces, not only the one-dimensional subspaces spanned by any of the direct products listed above. The state $|\phi\rangle$ is one of the multitude of arbitrary linear combinations of them.

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Historical remark

In their paper of 1935 Einstein, Podolsky and Rosen (EPR) proposed a thought experiment which was supposed to prove that QM is an *incomplete* theory. They used essentially the same *paradox* as the one discussed above.

They considered a pair of point particles with, for simplicity, one-dimensional coordinates x_1 and x_2 in the improper, i.e., unnormalized, state $|\phi\rangle$ described by the wave function

$$\langle x_1 x_2|\phi\rangle = \delta(x_1 - x_2 - a).$$

The Fourier transform of it has the form

$$\langle p_1 p_2|\phi\rangle = \frac{1}{2\pi}\int e^{-i(p_1x_1+p_2x_2)}\,\langle x_1x_2|\phi\rangle\,\mathrm{d}x_1\mathrm{d}x_2$$

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Although these two measurements on particle (2) cannot be performed simultaneously, neither of them acts directly on particle (1). From this EPR conclude that such measurements cannot *produce* the measured value x_1 and p_1 but merely *uncover* them. Therefore both the position and the momentum of particle (1) should have some kind of *physical reality*, independent of whether or not they are actually measured.

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Various forms of Bell's inequalities have been tested experimentally many times since 1972.

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