#### Bell's inequalities

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#### Introduction

By 1935, it was already recognized that the predictions of quantum mechanics (QM) are probabilistic. In their famous paper of 1935 Albert Einstein, Boris Podolsky and Nathan Rosen presented a scenario that, in their view, indicated that quantum particles, like electrons and photons, must carry physical properties or attributes not included in QM, and the uncertainties in predictions of QM were due to ignorance of these properties, later termed *hidden variables*.

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Consider a spin  $\frac{1}{2}$  particle. The spin operator has the form

 $\vec{S} = \frac{1}{2}\vec{\sigma},$ 

#### where we have assumed $\hbar = 1$ .

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Let  $\hat{a}$  denote the unit vector in the direction of the inhomogeneous magnetic field. Instead of spin projection onto it, i.e.,  $\hat{a} \cdot \vec{S}$  it is more convenient to use the observable

 $\dot{a}=2\hat{a}\cdot\vec{S},$ 

which has the eigenvalues  $\pm 1$  instead of  $\pm \frac{1}{2}$ .

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which has the eigenvalues  $\pm 1$  instead of  $\pm \frac{1}{2}$ .

If we choose  $\hat{a} = \hat{e}_3$  then we will obtain

$$\boldsymbol{e}_3 = 2\hat{\boldsymbol{e}}_3 \cdot \boldsymbol{\vec{S}} = \sigma_3,$$

with the eigenvectors  $\not{e}_3 \ket{\pm} = \pm \ket{\pm}$  of the following form

$$|+\rangle = \left( egin{array}{c} 1 \\ 0 \end{array} 
ight), \qquad |-\rangle = \left( egin{array}{c} 0 \\ 1 \end{array} 
ight).$$

Obviously

$$\langle +|$$
 = (1 0),  $\langle -|$  = (0 1).

Now, recall that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and calculate

$$\not = \hat{a} \cdot \vec{\sigma} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}.$$

Let us find the eigenvalues of a. To this end we have to solve the equation

$$\begin{vmatrix} a_3 - \lambda & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 - \lambda \end{vmatrix} = \lambda^2 - a_3^2 - a_1^2 - a_2^2 = 0.$$

As  $\hat{a}$  is the unit vector, we get  $\lambda = \pm 1$ .

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#### Thus, the eigenequation of a has the form

 $a |\hat{a}\pm\rangle = \pm |\hat{a}\pm\rangle$ .

Vector  $\hat{a}$  can be obtained from vector  $\hat{e}_3$  by a rotation of angle  $\vec{\theta} = \theta \hat{\theta}$ , with  $\hat{\theta}$  being a unit vector parallel to  $\hat{e}_3 \times \hat{a}$ , which determines the direction of the rotation axis. Hence, we have

$$\ket{\hat{a}\pm}=e^{-iec{ heta}\cdotec{S}}\ket{\pm},$$

where

$$e^{-i\vec{\theta}\cdot\vec{S}}S_3e^{i\vec{\theta}\cdot\vec{S}} = \hat{a}\cdot\vec{S}.$$

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Since it can be shown that

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Let us calculate matrix elements of a in the spin eigenvector basis

$$\langle +|\not a|+\rangle = (1 \ 0) \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \ 0) \begin{pmatrix} a_3 \\ a_1 + ia_2 \end{pmatrix}$$
$$= a_3,$$
$$\langle -|\not a|-\rangle = (0 \ 1) \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0 \ 1) \begin{pmatrix} a_1 - ia_2 \\ -a_3 \end{pmatrix}$$
$$= -a_3,$$
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$$= a_1 + ia_2.$$



Figure: Simultaneous spin measurements on particle pairs (1) + (2). S is the particle source, and  $\vec{a}$  and  $\vec{b}$  are field directions of the Stern–Gerlach magnets.

Now we consider the combination of two different spin- $\frac{1}{2}$  systems. A system of basis vectors is

$$egin{aligned} |(1)+
angle\otimes |(2)+
angle\,, & |(1)-
angle\otimes |(2)-
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# where (1) and (2) refer to the first and second particle, respectively, and + and - specifies the *z* component of the spin.

We can also use the Stern–Gerlach device that measures the spin component of the first particle along  $\hat{a}$  and the spin component of the second particle along  $\hat{b}$ .

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Then we take the following basis system

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with two arbitrarily chosen vectors unit vectors  $\hat{a}$  and  $\hat{b}$ .

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with two arbitrarily chosen vectors unit vectors  $\hat{a}$  and  $\hat{b}$ . Consider the following two observables

 $\mathbf{a} \otimes \mathbb{I} = 2\hat{\mathbf{a}} \cdot \vec{S} \otimes \mathbb{I}$  and  $\mathbb{I} \otimes \mathbf{b} = \mathbb{I} \otimes 2\hat{\mathbf{b}} \cdot \vec{S}$ ,

which are 2  $\times$  spin component of particle (1) along  $\hat{a}$  and 2  $\times$  spin component of particle (2) along  $\hat{b}$ , respectively.

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which are 2  $\times$  spin component of particle (1) along  $\hat{a}$  and 2  $\times$  spin component of particle (2) along  $\hat{b}$ , respectively.

They act on our basis vectors  $|\hat{a}\alpha\rangle \otimes |\hat{b}\beta\rangle$ ,  $\alpha, \beta = \pm 1$ , in the following way

$$\begin{split} \not{a} \otimes \mathbb{I} | \hat{a}\alpha \rangle \otimes | \hat{b}\beta \rangle &= \not{a} | \hat{a}\alpha \rangle \otimes \mathbb{I} | \hat{b}\beta \rangle = \alpha | \hat{a}\alpha \rangle \otimes | \hat{b}\beta \rangle \\ \mathbb{I} \otimes \not{b} | \hat{a}\alpha \rangle \otimes | \hat{b}\beta \rangle &= \mathbb{I} | \hat{a}\alpha \rangle \otimes \not{b} | \hat{b}\beta \rangle = \beta | \hat{a}\alpha \rangle \otimes | \hat{b}\beta \rangle. \end{split}$$

The two observables commute and our basis vectors are simultaneously eigenvectors of both of them with eigenvalues +1 or -1. Indeed, let us calculate

$$egin{aligned} [ \not a \otimes \mathbb{I}, \mathbb{I} \otimes eta ] &= ( \not a \otimes \mathbb{I} ) ( \mathbb{I} \otimes eta ) - ( \mathbb{I} \otimes eta ) ( \not a \otimes \mathbb{I} ) \ &= 
otag \mathbb{I} \otimes \mathbb{I} eta - \mathbb{I} 
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Thus  $\mathbf{a} \otimes \mathbb{I}$  and  $\mathbb{I} \otimes \mathbf{b}$  can be measured simultaneously.

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The measurement of  $\not a \otimes \mathbb{I}$  in the basis states

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will always yield +1, -1, +1, -1, and the simultaneous measurement of  $\mathbb{I} \otimes \mathbf{b}$  will yield +1, -1, -1, +1.

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will always yield +1, -1, +1, -1, and the simultaneous measurement of  $\mathbb{I} \otimes \mathbf{p}$  will yield +1, -1, -1, +1.

For such simultaneous measurement we can in addition define a *spin correlation observable*, which is by definition the product of the values obtained in a single measurement of both  $\not a \otimes \mathbb{I}$  and  $\mathbb{I} \otimes \not b$ .

This spin correlation observable is described by the operator

#### $a \otimes b$

whose eigenvalues values are +1 for the first two vectors and -1 for the last two vectors of the above basis.

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The simultaneous spin measurements on two-particle systems with two Stern–Gerlach devices are possible only if the two particles of each pair are spatially separated and each particle moves along a certain fixed axis, as shown in the Figure below.



Figure: Simultaneous spin measurements on particle pairs (1) + (2).

A particle source emits pairs of particles, one pair at a time, such that particle (1) is always emitted to the left, and particle (2) is always emitted to the right.

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Then a Stern–Gerlach device with inhomegenous magnetic field along some direction  $\hat{a}$ , perpendicular to the beam, may be applied to the left beam and another Stern–Gerlach device with field direction  $\hat{b}$  may be applied to the right beam. Each device has two counters, one at a position +1 and and the other at a position -1. Since the particles (1) and (2) are emitted pairwise by the source, the two particles of a single pair pass the two Stern–Gerlach magnets an arrive at two of the four counters almost simultaneously. Then a Stern–Gerlach device with inhomegenous magnetic field along some direction  $\hat{a}$ , perpendicular to the beam, may be applied to the left beam and another Stern–Gerlach device with field direction  $\hat{b}$  may be applied to the right beam. Each device has two counters, one at a position +1 and and the other at a position -1. Since the particles (1) and (2) are emitted pairwise by the source, the two particles of a single pair pass the two Stern–Gerlach magnets an arrive at two of the four counters almost simultaneously.

Therefore, a click of the +1 counter on the left and -1 counter on the right means a simultaneous measurement of  $\not a \otimes \mathbb{I}$  with the result +1 and of  $\mathbb{I} \otimes \not b$  with the result -1. The spin correlation observable  $\not a \otimes \not b$  has then the value (+1)(-1) = -1.

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This kind of measurement is repeated N times, with  $N \gg 1$ , and the following numbers are recorded:

- the number  $N_{++}$  of simultaneous clicks of +1 counter on the left and +1 counter on the right,
- the number  $N_{+-}$  of simultaneous clicks of +1 counter on the left and -1 counter on the right,

the numbers  $N_{-+}$  and  $N_{--}$  are defined similarly.

The measured *average values* for the observables  $\mathbf{a} \otimes \mathbb{I}$ ,  $\mathbb{I} \otimes \mathbf{b}$  and  $\mathbf{a} \otimes \mathbf{b}$ , which we denote, respectively, by  $E_1(\hat{a})$ ,  $E_2(\hat{b})$  and  $E(\hat{a}, \hat{b})$  are the following:

$$E_{1}(\hat{a}) = \frac{1}{N}(N_{++} + N_{+-} - N_{-+} - N_{--}),$$

$$E_{2}(\hat{b}) = \frac{1}{N}(N_{++} - N_{+-} + N_{-+} - N_{--}),$$

$$E(\hat{a}, \hat{b}) = \frac{1}{N}(N_{++} - N_{+-} - N_{-+} + N_{--}),$$
where obviously  $N = N_{++} + N_{+-} + N_{-+} + N_{--}.$ 

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where obviously  $N = N_{++} + N_{+-} + N_{-+} + N_{--}.$ 

According to quantum mechanics, these measured average values should coincide with the *expectation values* of corresponding observables in the common spin state of the particle pairs emitted by the source.

The combination of two spin- $\frac{1}{2}$  systems may lead to the total spin value

 $s = \left| \frac{1}{2} - \frac{1}{2} \right|, ..., \frac{1}{2} + \frac{1}{2},$ 

i.e., s = 0 or s = 1.

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We will assume here that the particle pairs emitted by the source have total spin 0, and are therefore in the asymmetric singlet state

$$egin{aligned} |\phi
angle &= rac{1}{\sqrt{2}}(|(1)+
angle\otimes|(2)-
angle-|(1)-
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angle) \ &\equiv rac{1}{\sqrt{2}}(|+
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A source emitting particle pairs in the spin state  $|\phi\rangle$  might contain, e.g., a large number of unstable compounds of two particles (1) and (2) at rest, and therefore after the decay, due to momentum conservation, particles (1) and (2) move always in opposite directions. However, while moving away, they are still in the common spin state  $|\phi\rangle$ .

We can now calculate the corresponding QM expectation values.

$$\begin{split} \langle \phi | \not = \frac{1}{2} (\langle +| \otimes \langle -| - \langle -| \otimes \langle +| \rangle \not = \mathbb{I}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle) \\ &= \frac{1}{2} (\langle +| \otimes \langle -| - \langle -| \otimes \langle +| \rangle (\not = |+\rangle \otimes |-\rangle - \not = |-\rangle \otimes |+\rangle) \\ &= \frac{1}{2} (\langle +| \not = |+\rangle \langle -|-\rangle - \langle +| \not = |-\rangle \langle -|+\rangle - \langle -| \not = |+\rangle \langle +|-\rangle \\ &+ \langle -| \not = |-\rangle \langle +|+\rangle) = \frac{1}{2} (a_3 - a_3) = 0, \end{split}$$

where we have used  $\langle +|\not a|+\rangle = a_3$  and  $\langle -|\not a|-\rangle = -a_3$ .
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angle \otimes | - 
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angle \langle - | - 
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Using  $\langle +|\not{a}|+\rangle = a_3$ ,  $\langle -|\not{a}|-\rangle = -a_3$ ,  $\langle +|\not{a}|-\rangle = a_1 - ia_2$  and  $\langle -|\not{a}|+\rangle = a_1 + ia_2$ , and analogously for  $\not{b}$ , we will get  $\langle \phi | \mathbb{I} \otimes \not{b} | \phi \rangle = \frac{1}{2} (\langle +| \otimes \langle -| - \langle -| \otimes \langle +| \rangle) \mathbb{I} \otimes \not{b} (|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$   $= \frac{1}{2} (\langle +| \otimes \langle -| - \langle -| \otimes \langle +| \rangle) (|+\rangle \otimes \not{b} |-\rangle - |-\rangle \otimes \not{b} |+\rangle)$   $= \frac{1}{2} (\langle +|+\rangle \langle -|\not{b}|-\rangle - \langle +|-\rangle \langle -|\not{b}|+\rangle - \langle -|+\rangle \langle +|\not{b}|-\rangle$  $+ \langle -|-\rangle \langle +|\not{b}|+\rangle) = \frac{1}{2} (-b_3 + b_3) = 0.$ 

Similarly

$$\begin{split} \langle \phi | \not a \otimes \not b | \phi \rangle &= \frac{1}{2} (\langle + | \not a | + \rangle \langle - | \not b | - \rangle - \langle + | \not a | - \rangle \langle - | \not b | + \rangle - \langle - | \not a | + \rangle \langle + | \not b | - \rangle \\ &+ \langle - | \not a | - \rangle \langle + | \not b | + \rangle) = \frac{1}{2} (a_3(-b_3) - (a_1 - ia_2)(b_1 + ib_2) \\ &- (a_1 + ia_2)(b_1 - ib_2) + (-a_3)b_3) = -\hat{a} \cdot \hat{b}. \end{split}$$

The QM predictions for the expectation values  $\langle \phi | \mathbf{a} \otimes \mathbb{I} | \phi \rangle$ ,  $\langle \phi | \mathbb{I} \otimes \mathbf{b} | \phi \rangle$  and  $\langle \phi | \mathbf{a} \otimes \mathbf{b} | \phi \rangle$  hold obviously for a large number of single particle pair spin measurements.

Compare the QM prediction for

 $\langle \phi | \mathbf{a} \otimes \mathbb{I} | \phi 
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with the measured average value

$$E_1(\hat{a}) = \frac{1}{N}(N_{++} + N_{+-} - N_{-+} - N_{--}).$$

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We see that the QM prediction implies

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which means that the number of cases in which the spin of particle (1) is found to be parallel and atiparallel to  $\hat{a}$  are equal for any choice of  $\hat{a}$ . This result is a consequence of the rotational invariance of the spin state  $|\phi\rangle$ .

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The same conclusion can be derived for particle (2) if we compare the QM prediction for  $\langle \phi | \mathbb{I} \otimes \mathbf{p} | \phi \rangle$  with the measured average value of

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At the first glance comparison of the QM prediction for the spin correlation  $\langle \phi | \not a \otimes \not b | \phi \rangle$  with the measured average value of

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Then, remembering that  $N = N_{++} + N_{+-} + N_{-+} + N_{--}$  we will get the following condition

 $N_{++} - N_{+-} - N_{-+} + N_{--} = -N_{++} - N_{+-} - N_{-+} - N_{--}$ 

and, since both  $N_{++}$  and  $N_{--}$  are positive, we see that

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According to the QM prediction

 $E(\hat{a},\hat{a})=-1$ 

spin components of two particles (1) and (2) along a fixed direction  $\hat{a}$  are always opposite to each other.

Instead of directly measuring  $\not a \otimes \mathbb{I}$  on particle (1) itself, we can equally well determine its spin component along  $\hat{a}$  by measuring  $\mathbb{I} \otimes \not a$  on particle (2) and multiplying the result by -1.

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Therefore, it seems quite *natural* to imagine that a single particle (1) does not *get* a definite spin component along  $\hat{a}$  during a measurement of  $\hat{a} \otimes \mathbb{I}$ , but rather *has* a definite value of it, either +1 or -1, prior to and independent of any measurement.

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Assume that prior to and independent of any measurement every single particle (1) possesses a definite value  $v(\hat{a})$ , of either +1 or -1, for the components of its spin, at least along all possible directions  $\hat{a}$  orthogonal to the beam.

These values are just *uncovered*, rather than *produced*, if the actual spin measurement is performed. They may be visualized as hidden labels, either +1 or -1, attached to every single particle (1) for every possible direction  $\hat{a}$ . The same argument applies obviously to all particles (2).

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Without this assumption it seems quite difficult to understand the perfect anticorrelation  $\langle \phi | \not a \otimes \not a | \phi \rangle = -1$  for simultaneous measurements of  $\not a \otimes \mathbb{I}$  on particle (1) and  $\mathbb{I} \otimes \not a$  on particle (2).

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Consider a very large number N of particle pairs in the spin singlet state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$$

and four arbitrarily chosen directions  $\hat{a}$ ,  $\hat{b}$ ,  $\hat{c}$  and  $\hat{d}$  in the plane orthogonal to the two beams of particles produced by the source. Denote by  $v_i(\hat{a})$  and  $v_i(\hat{d})$  the *hidden* predetermined values of the spin components along  $\hat{a}$  and  $\hat{d}$  of particle (1) in the *i*-th pair, and by  $w_i(\hat{b})$  and  $w_i(\hat{c})$  the *hidden* predetermined values of the spin components along  $\hat{b}$  and  $\hat{c}$  of particle (2) in the same pair.

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If such experiment had been performed instead with the same N particle pairs, it would have uncovered the spin components  $v_i(\hat{d})$  and  $w_i(\hat{c})$  and the observed spin correlation average would have been

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It seems reasonable to assume that an experimental determination of  $E(\hat{d}, \hat{c})$  would have produced the same result if it had been performed on any of the four sets of N particle pairs.

Therefore  $E(\hat{a}, \hat{b})$ ,  $E(\hat{d}, \hat{c})$ ,  $E(\hat{a}, \hat{c})$  and  $E(\hat{d}, \hat{b})$  may be considered as quantities characteristic of the *N* particle pairs set and their preparation, and are not dependent on which particular experiment is actually performed.

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Although we know the QM predictions for all all of them, i.e.,  $-\hat{a} \cdot \hat{b}$ ,  $-\hat{d} \cdot \hat{c}$ ,  $-\hat{a} \cdot \hat{c}$  and  $-\hat{d} \cdot \hat{b}$ , respectively, we will not yet assume this, but we will use the *experimental* results given of the previous slide.

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 $E(\hat{a},\hat{b})+E(\hat{a},\hat{c})+E(\hat{d},\hat{b})-E(\hat{d},\hat{c}).$ 

To this end let us first show that

 $v_i(\hat{a})(w_i(\hat{b})+w_i(\hat{c}))+v_i(\hat{d})(w_i(\hat{b})-w_i(\hat{c}))=\pm 2,$ 

i = 1, 2, ..., N. **Proof.** As  $w_i$  is either +1 or -1, the first bracket is either +2, 0 or -2.

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$$-2N\leq \sum_{i=1}^{N}\left[v_i(\hat{a})(w_i(\hat{b})+w_i(\hat{c}))+v_i(\hat{d})(w_i(\hat{b})-w_i(\hat{c}))
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and dividing this by N we obtain

$$\begin{split} \frac{1}{N}\sum_{i=1}^{N}v_i(\hat{a})w_i(\hat{b}) + \frac{1}{N}\sum_{i=1}^{N}v_i(\hat{a})w_i(\hat{c}) + \frac{1}{N}\sum_{i=1}^{N}v_i(\hat{d})w_i(\hat{b}) \\ & -\frac{1}{N}\sum_{i=1}^{N}v_i(\hat{d})w_i(\hat{c}) \bigg| \leq 2. \end{split}$$

Thus, we obtain the inequality

$$\left| E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{c}) + E(\hat{d}, \hat{b}) - E(\hat{d}, \hat{c}) \right| \leq 2,$$

which is the most famous and experimentally most useful of a series of similar inequalities known as *Bell's inequalities*.

Let us now check if the QM prediction

 $E(\hat{a},\hat{b}) = \langle \phi | \not a \otimes \not b | \phi 
angle = -\hat{a} \cdot \hat{b}$ 

satisfies the above Bell's inequality.

$$\begin{aligned} |-\hat{a}\cdot\hat{b}-\hat{a}\cdot\hat{c}-\hat{d}\cdot\hat{b}+\hat{d}\cdot\hat{c}| &= |\hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c}+\hat{d}\cdot\hat{b}-\hat{d}\cdot\hat{c}| \\ &= |\hat{a}\cdot(\hat{b}+\hat{c})+\hat{d}\cdot(\hat{b}-\hat{c})| \leq |\hat{a}||\hat{b}+\hat{c}|+|\hat{d}||\hat{b}-\hat{c}| \\ &= |\hat{b}+\hat{c}|+|\hat{b}-\hat{c}| = \sqrt{(\hat{b}+\hat{c})^2} + \sqrt{(\hat{b}-\hat{c})^2} \\ &= \sqrt{2+2\cos\theta} + \sqrt{2-2\cos\theta}, \end{aligned}$$

with  $\theta$  being the angle between  $\hat{b}$  and  $\hat{c}$ ,  $\hat{b} \cdot \hat{c} = \cos \theta$ ,  $\theta \in [0, \pi]$ .

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with  $\theta$  being the angle between  $\hat{b}$  and  $\hat{c}$ ,  $\hat{b} \cdot \hat{c} = \cos \theta$ ,  $\theta \in [0, \pi]$ .

Denote the expression on the right hand side of our inequality by

$$f(\theta) = \sqrt{2 + 2\cos\theta} + \sqrt{2 - 2\cos\theta} = 2\sqrt{\frac{1 + \cos\theta}{2}} + 2\sqrt{\frac{1 - \cos\theta}{2}},$$

which for  $\theta \in [0,\pi]$  can be written as

$$f(\theta) = 2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}$$

Let us find the maximum of  $f(\theta)$ .

$$f'( heta) = -\sinrac{ heta}{2} + \cosrac{ heta}{2} = 0 \quad \Leftrightarrow \quad rac{ heta}{2} = rac{\pi}{4}.$$

Thus  $f'(\theta) = 0$  for  $\theta = \frac{\pi}{2}$ . Calculate

$$f''(\theta) = -\frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{2}\sin\frac{\theta}{2}\Big|_{\theta = \frac{\pi}{2}} = -2\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2} = -2\sqrt{2} < 0.$$

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Thus,  $f(\theta)$  has the maximum at  $\theta = \frac{\pi}{2}$  equal to

$$f(\theta) = \sqrt{2 + 2\cos{\frac{\pi}{2}}} + \sqrt{2 - 2\cos{\frac{\pi}{2}}} = 2\sqrt{2}$$

and the QM prediction for our inequality is the following

 $\left| \mathsf{E}(\hat{\mathsf{a}},\hat{b}) + \mathsf{E}(\hat{\mathsf{a}},\hat{c}) + \mathsf{E}(\hat{d},\hat{b}) - \mathsf{E}(\hat{d},\hat{c}) \right| \leq 2\sqrt{2}.$ 

The Bell inequality becomes equality, i.e., it is maximally violated by the QM prediction, if

- ()  $\hat{a}$  and  $\hat{b} + \hat{c}$ , and  $\hat{d}$  and  $\hat{b} \hat{c}$  are parallel,
- 2)  $\hat{a}$  and  $\hat{b} + \hat{c}$ , and  $\hat{d}$  and  $\hat{b} \hat{c}$  are antiparallel.

These configurations are depicted in the Figure below.



Figure: Magnetic field configurations of the Stern–Gerlach devices for which Bell's inequality is maximally violated.

$$\left| E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{c}) + E(\hat{d}, \hat{b}) - E(\hat{d}, \hat{c}) \right| = 2\sqrt{2}$$

clearly contradicts Bell's inequality

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There are infinitely many configurations of directions  $\hat{a}$ ,  $\hat{b}$ ,  $\hat{c}$  and  $\hat{d}$  for which the QM predictions **do not** satisfy Bell's inequality.

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This is not as easy in practice as one might imagine, however. First of all it is difficult to prepare pairs of spin- $\frac{1}{2}$  particles in a spin singlet state  $|\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$ .

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Figure: Scheme of a photon analyzer for tests of Bell's inequalities.

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For the photon pairs emitted in cascade transitions, these polarizations are correlated in much the same way as the spin components of spin- $\frac{1}{2}$  particle pairs in the considered singlet state  $|\phi\rangle$ .

However, the QM prediction for  $E(\hat{a}, \hat{b})$  is different and therefore the configurations of directions  $\hat{a}$ ,  $\hat{b}$ ,  $\hat{c}$  and  $\hat{d}$  for which Bell's inequality

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The hypothesis that two spin- $\frac{1}{2}$  particles *have* definite spin components  $v_i(\hat{a})$  and  $w_i(\hat{b})$  prior to the measurement seems natural only from the classical point of view, in which we visualize the particle pair as consisting of two separate particles.

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However, the QM state vector

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angle = rac{1}{\sqrt{2}}(|(1)+
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angle-|(1)-
angle\otimes|(2)+
angle)$$

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The QM state vector  $|\phi\rangle$  describes a new entity, an *indivisible whole*, a single object whose constituent particles (1) and (2) are not definable until a measurement is made that prepares the direct product states  $|(1)+\rangle \otimes |(2)+\rangle$ ,  $|(1)-\rangle \otimes |(2)-\rangle$ ,  $|(1)+\rangle \otimes |(2)-\rangle$  and  $|(1)-\rangle \otimes |(2)+\rangle$ , or their mixtures. Being a state with the total spin 0,  $|\phi\rangle$  does not have

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They considered a pair of point particles with, for simplicity, one-dimensional coordinates  $x_1$  and  $x_2$  in the improper, i.e., unnormalized, state  $|\phi\rangle$  described by the wave function

 $\langle x_1x_2|\phi\rangle = \delta(x_1-x_2-a).$ 

The Fourier transform of it has the form

$$\begin{split} \langle p_1 p_2 | \phi \rangle &= \frac{1}{2\pi} \int e^{-i(p_1 x_1 + p_2 x_2)} \langle x_1 x_2 | \phi \rangle \, dx_1 dx_2 \\ &= \frac{1}{2\pi} \int e^{-i(p_1 x_1 + p_2 x_2)} \delta(x_1 - x_2 - a) dx_1 dx_2 = \frac{1}{2\pi} \int e^{-i(p_1 (x_2 + a) + p_2 x_2)} dx_2 \\ &= e^{-ip_1 a} \frac{1}{2\pi} \int e^{-i(p_1 + p_2) x_2} dx_2 = e^{-ip_1 a} \delta(p_1 + p_2), \end{split}$$

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simultaneous measurements of the positions of the two particles always yield two values  $x_1$  and  $x_2$  related by  $x_1 = x_2 + a$  and according to

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Although these two measurements on particle (2) cannot be performed simultaneously, neither of them acts directly on particle (1). From this EPR conclude that such measurements cannot produce the measured value  $x_1$  and  $p_1$  but merely uncover them. Therefore both the position and the momentum of particle (1) should have some kind of physical reality, independent of whether or not they are actually measured.

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